

Three New Results on Continuation Criteria for the 3D Relativistic Vlasov-Maxwell System

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Abstract

In this paper, we consider sufficient conditions, called continuation criteria, for global existence and uniqueness of classical solutions to the three-dimensional relativistic Vlasov-Maxwell system. In the compact momentum support setting, we prove that $\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$, where $1 \leq r \leq 2$ and $\beta > 0$ is arbitrarily small, is a continuation criteria. The previously best known continuation criteria in the compact setting is $\|p_0^{\frac{4}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$, where $1 \leq r < \infty$ and $\beta > 0$ is arbitrarily small, is due to Kunze in [7]. Thus, our continuation criteria is an improvement in the $1 \leq r \leq 2$ range. In addition, we consider also sufficient conditions for a global existence result to the three-dimensional relativistic Vlasov-Maxwell system without compact support in momentum space, building on previous work by Luk-Strain [9]. In [9], it was shown that $\|p_0^\theta f\|_{L_x^1 L_p^1} \lesssim 1$ is a continuation criteria for the relativistic Vlasov-Maxwell system without compact support in momentum space for $\theta > 5$. We improve this result to $\theta > 3$. We also build on another result by Luk-Strain in [8], in which the authors proved the existence of a global classical solution in the compact momentum support setting given the condition that there exists a two-dimensional plane on which the momentum support of the particle density remains fixed. We prove well-posedness even if the plane varies continuously in time.

1 Introduction

Consider a distribution of charged particles described by a non-negative density function $f : \mathbb{R}_t \times \mathbb{R}_x^3 \times \mathbb{R}_p^3 \rightarrow \mathbb{R}_+$ of time t , space x and momentum p . The Vlasov-Maxwell system describes the evolution of the density function $f(t, x, p)$ under the influence of time-dependent vector fields $E, B : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}^3$. Physically, this system models the behavior of a collisionless plasma:

$$\partial_t f + \hat{p} \cdot \nabla_x f + (E + \hat{p} \times B) \cdot \nabla_p f = 0, \quad (1)$$

$$\partial_t E = \nabla_x \times B - j, \quad \partial_t B = -\nabla_x \times E, \quad (2)$$

$$\nabla_x \cdot E = \rho, \quad \nabla_x \cdot B = 0. \quad (3)$$

Here the charge is

$$\rho(t, x) \stackrel{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} f(t, x, p) dp,$$

and the current is given by

$$j_i(t, x) \stackrel{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} \hat{p}_i f(t, x, p) dp, \quad i = 1, \dots, 3$$

with initial data $(f, E, B)|_{t=0} = (f_0, E_0, B_0)$ satisfying the time-independent equations (3). In the above equations, $\hat{p} = \frac{p}{p_0}$ where $p_0 = (1 + |p|^2)^{\frac{1}{2}}$.

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1.1 Notation

In this section, we describe the notation that will be followed in the remainder of this paper. For a scalar function, $f = f(t, x, p)$, and real numbers $1 \leq s, r, q \leq \infty$ we define the following norm:

$$\|f\|_{L^s([0,T]; L_x^r L_p^q)} \stackrel{\text{def}}{=} \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f|^q dp \right)^{\frac{r}{q}} dx \right)^{\frac{s}{r}} dt \right)^{\frac{1}{s}}.$$

Next, we define $K = E + \hat{p} \times B$ and note that $|K| \leq |E| + |B|$ since $|\hat{p}| = 1$. Using the notation from [7], we define:

$$\sigma_{-1}(t, x) \stackrel{\text{def}}{=} \sup_{|\omega|=1} \int_{\mathbb{R}^3} \frac{f(t, x, p)}{p_0(1 + \hat{p} \cdot \omega)} dp.$$

Also, for use in the case where we have control of the momentum support of f , we define

$$P(t) \stackrel{\text{def}}{=} 2 + \sup\{p \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3, s \in [0, t] \text{ such that } f(s, x, p) \neq 0\}. \quad (4)$$

The notation $a \lesssim b$ means that there exists some positive inessential constant, C , such that $a \leq Cb$ and $a \approx b$ means that $\frac{1}{C}b \leq a \leq Cb$.

Next, define the integral over the space-time cone $C_{t,x}$ as follows:

$$\int_{C_{t,x}} f d\sigma \stackrel{\text{def}}{=} \int_0^t \int_0^{2\pi} \int_0^\pi (t-s)^2 \sin(\theta) f(s, x + (t-s)\omega) d\theta d\phi ds \quad (5)$$

in which $\omega = (\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta))$.

Finally, for a plane $Q \subset \mathbb{R}^3$ containing the origin we define the projection \mathbb{P}_Q to be the orthogonal projection onto the plane Q .

1.2 Preliminaries

By the method of characteristics, we obtain that the particle density is conserved over the characteristics described by the system of ordinary differential equations:

$$\frac{dX}{ds}(s; t, x, p) = \hat{V}(s; t, x, p), \quad (6)$$

$$\frac{dV}{ds}(s; t, x, p) = E(s, X(s; t, x, p)) + \hat{V}(s; t, x, p) \times B(s, X(s; t, x, p)), \quad (7)$$

together with the conditions

$$X(t; t, x, p) = x, \quad V(t; t, x, p) = p, \quad (8)$$

where $\hat{V} \stackrel{\text{def}}{=} \frac{V}{\sqrt{1+|V|^2}}$. Further, we also have the conservation laws:

Proposition 1. *Suppose (f, E, B) is a solution to the relativistic Vlasov-Maxwell system. Then we have the following conservation laws:*

$$\frac{1}{2} \int_{\{t\} \times \mathbb{R}^3} (|E|^2 + |B|^2) dx + 4\pi \int_{\{t\} \times \mathbb{R}^3 \times \mathbb{R}^3} p_0 f(t, x, p) dx dp = \text{constant} \quad (9)$$

and

$$\|f\|_{L_{x,p}^\infty}(t) \leq \|f_0\|_{L_{x,p}^\infty}. \quad (10)$$

Note that by interpolation, the conservation laws above imply that $\|f\|_{L_{x,p}^q}(t) \lesssim \|f_0\|_{L_{x,p}^q}$ for $1 \leq q \leq \infty$. Thus, given sufficiently nice initial data, we can assume L^2 bounds on K and L^q bounds on f for $1 \leq q \leq \infty$. The Glassey-Strauss decomposition is $E = E_0 + E_T + E_S$, where E_0 depends only on the initial data, and E_T and E_S are:

$$E_T = - \int_{|y-x| \leq t} \int_{\mathbb{R}^3} \frac{(\omega + \hat{p})(1 - |\hat{p}|^2)}{(1 + \hat{p} \cdot \omega)^2} f(t - |y - x|, y, p) dp \frac{dy}{|y - x|^2} \quad (11)$$

$$E_S = - \int_{|y-x| \leq t} \int_{\mathbb{R}^3} \nabla_p \left(\frac{(\omega + \hat{p})}{1 + \hat{p} \cdot \omega} \right) \cdot K f dp \frac{dy}{|y-x|} \quad (12)$$

The Glassey-Strauss decomposition for the magnetic field $B = B_0 + B_T + B_S$ is similar. Writing $K_0 = E_0 + \hat{p} \times B_0$, $K_T = E_T + \hat{p} \times B_T$ and $K_S = E_S + \hat{p} \times B_S$, we can write that $|K| \leq |E| + |B|$, $|K_0| \leq |E_0| + |B_0|$, $|K_T| \leq |E_T| + |B_T|$, and $|K_S| \leq |E_S| + |B_S|$, where K_0 depends only on the initial data (f_0, E_0, B_0) of the relativistic Vlasov-Maxwell system. Bounding the other terms as in Propositions 3.1 and 3.2 in [8] we obtain:

$$|K_T| \lesssim \int_{|y| \leq t} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} \frac{f(t - |y|, x + y, p)}{p_0(1 + \hat{p} \cdot \omega)} dp$$

and

$$|K_S| \lesssim \int_{|y| \leq t} \frac{dy}{|y|} \int_{\mathbb{R}^3} \frac{(|E| + |B|) f(t - |y|, x + y, p)}{p_0(1 + \hat{p} \cdot \omega)} dp$$

Recalling the definition of σ_{-1} , we can bound these expressions by:

$$|K_T| \lesssim \int_{|y| \leq t} \frac{\sigma_{-1}(t - |y|, x + y) dy}{|y|^2} \quad (13)$$

$$|K_S| \lesssim \int_{|y| \leq t} \frac{(|E| + |B|) \sigma_{-1}(t - |y|, x + y) dy}{|y|} \quad (14)$$

Note that the right hand side of (14) is in the form of $\square^{-1}(|K| \sigma_{-1})$, where $\square \stackrel{\text{def}}{=} \partial_t^2 - \sum_{i=1}^3 \partial_{x_i}^2$ and $u = \square^{-1} F$ satisfies:

$$\square u = F; \quad u|_{t=0} = \partial_t u|_{t=0} = 0 \quad (15)$$

1.3 Previous Results

Luk-Strain [8] stated the following version of the Glassey-Strauss result in [4] in the case where f_0 is compactly supported in momentum space:

Theorem 2. *Consider initial data (f_0, E_0, B_0) where $f_0 \in H^5(\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ is non-negative and has compact support in (x, p) , and $E_0, B_0 \in H^5(\mathbb{R}_x^3)$ such that (3) holds. Suppose (f, E, B) is the unique classical solution to the relativistic Vlasov-Maxwell system (1) – (3) in the time interval $[0, T)$ and there exists a bounded continuous function $P : [0, T) \rightarrow \mathbb{R}_+$ such that*

$$f(t, x, p) = 0 \text{ for } |p| \geq P(t) \quad \forall x \in \mathbb{R}, t \in [0, T).$$

Then our solution (f, E, B) extends uniquely in C^1 to a larger time interval $[0, T + \epsilon]$ for some $\epsilon > 0$.

Additional assumptions on the Vlasov-Maxwell system, such as the condition that $f(t, x, p) = 0$ for $|p| \geq P(t) \quad \forall x \in \mathbb{R}, t \in [0, T)$ in the above theorem, are known as continuation criteria as they allow us to extend the interval of existence of a solution. Luk-Strain [9] removed the condition of compact support in momentum space and proved continuation criteria in the space $H^D(w_3(p)^2 \mathbb{R}_x^3 \times \mathbb{R}_p^3)$, which is the weighted Sobolev space defined by the norm:

$$\|f\|_{H^D(w_3(p)^2 \mathbb{R}_x^3 \times \mathbb{R}_p^3)} = \sum_{0 \leq k \leq D} \|(\nabla_{x,p}^k f) w_3\|_{L_x^2 L_p^2}$$

where the weight is defined as $w_3(p) = p_0^{\frac{3}{2}} \log(1 + p_0)$. Luk-Strain [9] proved the following in this weighted Sobolev space:

Theorem 3. *Let $(f_0(x, p), E_0(x), B_0(x))$ be a 3D initial data set which satisfies the constraints (3) and such that for some $D \geq 4$, $f_0 \in H^D(w_3(p)^2 \mathbb{R}_x^3 \times \mathbb{R}_p^3)$ is non-negative and obeys the bounds*

$$\sum_{0 \leq k \leq D} \|(\nabla_{x,p}^k f_0) w_3\|_{L_x^2 L_p^2} < \infty, \quad (16)$$

$$\left\| \int_{\mathbb{R}^3} \sup\{f_0(x+y, p+w)p_0^3 : |y|+|w| \leq R\} dp \right\|_{L_x^\infty} \leq C_R, \quad (17)$$

$$\left\| \int_{\mathbb{R}^3} \sup\{|\nabla_{x,p} f_0|(x+y, p+w)p_0^3 : |y|+|w| \leq R\} dp \right\|_{L_x^\infty} \leq C_R, \quad (18)$$

$$\left\| \int_{\mathbb{R}^3} \sup\{|\nabla_{x,p} f_0|^2(x+y, p+w)w_3^2 : |y|+|w| \leq R\} dp \right\|_{L_x^\infty} \leq C_R^2, \quad (19)$$

and

$$\left\| \int_{\mathbb{R}^3} \sup\{|\nabla_{x,p}^2 f_0|(x+y, p+w)p_0 : |y|+|w| \leq R\} dp \right\|_{L_x^\infty} \leq C_R, \quad (20)$$

for some different constants $C_R < \infty$ for every $R > 0$; and the initial electromagnetic fields $E_0, B_0 \in H^D(\mathbb{R}_x^3)$ obey the bounds

$$\sum_{0 \leq k \leq D} (\|\nabla_x^k E_0\|_{L_x^2} + \|\nabla_x^k B_0\|_{L_x^2}) < \infty. \quad (21)$$

Given this initial data set, there exists a unique local solution (f, E, B) on some $[0, T_{loc}]$ such that $f_0 \in L^\infty([0, T_{loc}]; H^4(w_3(p)^2 \mathbb{R}_x^3 \times \mathbb{R}_p^3))$ and $E, B \in L^\infty([0, T_{loc}]; H^4(\mathbb{R}_x^3))$.

Let (f, E, B) be the unique solution to (1)-(3) in $[0, T_*)$. Assume that

$$\sup \int_0^{T_*} (|E(s, X(s; t, x, p))| + |B(s, X(s; t, x, p))|) ds < \infty \quad (22)$$

where the supremum is taken over all $(t, x, p) \in [0, T_*) \times \mathbb{R}^3 \times \mathbb{R}^3$. Then, there exists $\epsilon > 0$ such that the solution extends uniquely beyond T_* to an interval $[0, T_* + \epsilon]$ such that $E, B \in L^\infty([0, T_* + \epsilon]; H^D(\mathbb{R}_x^3))$ and $f \in L^\infty([0, T_* + \epsilon]; H^D(w_3(p)^2 dp dx))$.

Under the additional assumption that $\|p_0^N f_0\|_{L_t^\infty([0, T_*]; L_x^1 L_p^1)} \lesssim C_N$ for a large positive integer $N = N_\theta$ depending on θ , Luk-Strain [9] use the above theorem to prove that $\|p_0^\theta f\|_{L_t^\infty([0, T_*]; L_x^1 L_p^1)} \lesssim 1$ is a continuation criteria for the relativistic Vlasov-Maxwell system without compact support in momentum space for $\theta > 5$. To do so, Luk-Strain [9] utilized Strichartz estimates on both the K_T and K_S bounds and interpolation inequalities. We note in this paper that we only need the initial data assumption that $\|p_0^N f_0\|_{L_t^\infty([0, T_*]; L_x^1 L_p^1)} \lesssim 1$ for some $N > 5$. Finally, for comparison to the results in this paper, we also present the result in the compact support setting due to Kunze in [7]:

Theorem 4. Suppose we have initial data $f_0 \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $E_0, B_0 \in C_0^2(\mathbb{R}^3)$ satisfying the constraints (3). Let (f, E, B) be the unique solution to (1)-(3) in the time interval $[0, T_*]$. If

$$\|p_0^{\frac{4}{3}-1+\beta} f\|_{L^\infty([0, T_*]; L_x^r L_p^1)} < \infty$$

for some $1 \leq q < \infty$ and $\beta > 0$, then we can continuously extend our solution (f, E, B) uniquely to an interval $[0, T_* + \epsilon]$.

The final result of this paper builds on the work of Luk-Strain[8] in the compact setting:

Theorem 5. Consider initial data (f_0, E_0, B_0) where $f_0 \in H^5(\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ is non-negative and has compact support in (x, p) , and $E_0, B_0 \in H^5(\mathbb{R}_x^3)$ such that (3) holds. Suppose (f, E, B) is the unique classical solution to the relativistic Vlasov-Maxwell system (1) – (3) in the time interval $[0, T)$. Assume that there exists a plane $Q \subset \mathbb{R}^3$ with $0 \in Q$ and a bounded continuous function $\kappa : [0, T_+) \rightarrow \mathbb{R}^3$ such that

$$f(t, x, p) = 0 \quad \text{for } |\mathbb{P}_Q p| \geq \kappa(t), \quad \forall x \in \mathbb{R}^3.$$

Then there exists an $\epsilon > 0$ such that the solution extends uniquely in C^1 to a larger time interval $[0, T + \epsilon]$.

1.4 Main Results

We extend the result of Luk-Strain [9] to the case where $\theta > 3$ in Theorem 6 below. Note that we also remove the θ dependence of N in the moment bound, $\|p_0^N f_0\|_{L_t^\infty([0, T_*]; L_x^1 L_p^1)} \lesssim C_N$, of the initial data.

Theorem 6. *Consider initial data (f_0, E_0, B_0) satisfying (16)-(21) and the additional condition that $\|p_0^{\tilde{N}} f_0\|_{L_t^\infty([0, T_*]; L_x^1 L_p^1)} \lesssim C_N$ for some $\tilde{N} > 5$. Let (f, E, B) be the unique solution to (1) – (3) in $[0, T]$ and assume that*

$$\|p_0^\theta f\|_{L_x^1 L_p^1}(t) \leq A(t)$$

for some $\theta > 3$ and some bounded continuous function $A : [0, T] \rightarrow \mathbb{R}_+$. Then we can extend our solution (f, E, B) uniquely to an interval $[0, T + \epsilon]$ such that $E, B \in L^\infty([0, T + \epsilon]; H^D(\mathbb{R}_x^3))$ and $f \in L^\infty([0, T + \epsilon]; H^D(w_3(p)^2 dp dx))$.

1.4.1 Outline of Proof

The key to our proof is to gain bounds on $\|p_0^N f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)}$ for a power of $N > 5$ since we later prove that the expression in (22) can be bounded by $\|p_0^N f\|_{L_x^1 L_p^1}$ where $N = 5 + \lambda$ for any $\lambda > 0$. As proven in Proposition 7.3 in Luk-Strain [9], we have the following standard moment estimate for $N > 0$:

$$\|p_0^N f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)} \lesssim \|p_0^N f_0\|_{L_x^1 L_p^1} + \|E\|_{L_t^1([0, T]; L_x^{N+3})}^{N+3} + \|B\|_{L_t^1([0, T]; L_x^{N+3})}^{N+3} \quad (23)$$

Assume $N > 3$. Our goal now is to bound the terms $\|E\|_{L_t^1([0, T]; L_x^{N+3})}^{N+3}$ and $\|B\|_{L_t^1([0, T]; L_x^{N+3})}^{N+3}$ on the right hand side by $\|p_0^N f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)}^\alpha$ for some $\alpha < 1$. To do so, we employ the Glassey-Strauss decomposition of the field term

$$\tilde{K} \stackrel{\text{def}}{=} (E, B) = (E_0, B_0) + (E_T, B_T) + (E_S, B_S) \quad (24)$$

where E_0 and B_0 depend only on the initial data of our system. The terms on the right hand side of (24) have the same bounds as the K_T and K_S bounds in (13) and (14) respectively. To bound the K_T term, we utilize estimates for the averaging operator on the sphere and then apply the interpolation inequality used in Luk-Strain [9]. To do so, we define the operator

$$W_\alpha(h(t, x)) \stackrel{\text{def}}{=} \int_0^t s^{2-\alpha} \oint_{\mathbb{S}^2} h(t-s, x+s\omega) d\mu(\omega) ds,$$

where $\oint_X g(x) d\mu(x)$ denotes the average value of the function g over the measure space (X, μ) . We can bound K_T with this operator setting $\alpha = 2$:

$$K_T \lesssim W_2(\sigma_{-1}).$$

Thus, using a known averaging operator estimate and interpolation inequalities, we obtain the following bound on K_T :

$$\|K_T\|_{L_t^r([0, T]; L_x^{N+3})}^{N+3} \lesssim \|p_0\|_{L_t^r([0, T]; L_x^{N+3})}^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} \|f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)}^{2+\gamma} \|p_0^N f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)}^{1-\gamma}$$

for $1 \leq r \leq \infty$, $\gamma \in (0, 1)$, $N > 3$ and $\delta > 0$. To bound the K_S term, we apply Strichartz estimates for the wave equation and utilize the method from Sogge [12] as used in Kunze [7]. This method requires us to use the assumption that

$$\|\sigma_{-1}\|_{L_t^\infty([0, T]; L_x^2)} \lesssim 1.$$

We apply wave equation Strichartz estimates on a partition of the interval $[0, T] = \cup_{i=1}^{k-1} [T_i, T_{i+1}]$ such that the quantity $\|\sigma_{-1}\|_{L_t^\infty([T_i, T_{i+1}]; L_x^2)}$ is sufficiently small for us to use an iteration scheme to bound K_S over the interval $[0, T]$.

In Luk-Strain [9], the K_T term was bounded by using Hölder's inequality to rewrite the bound (14) in the form of a solution to the wave equation described by (15). Then, they used Strichartz estimates for the wave equation. Instead, we use a more direct approach by using averaging operator estimates. This approach also enables us to preserve the singularity in the σ_{-1} denominator, which is useful in reducing the power of p_0 in the bound of K_T . By the bounds on K_T and K_S , we obtain a bound on $\|K\|_{L_t^1([0, T]; L_x^{N+3})}^{N+3}$ for some $\gamma \in (0, 1)$:

$$\|K\|_{L_t^1([0,T];L_x^{N+3})}^{N+3} \lesssim 1 + \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^{2+\gamma} \|p_0^N f\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^{1-\gamma}$$

where the implicit constant in this inequality also depends on the quantity $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)}$.

Thus, assuming that $\|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$, we can insert this estimate into the standard moment estimate above to gain a higher moment bound $\|p_0^N f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$. By an iteration of this process, we eventually arrive at the bound $\|p_0^{\hat{N}} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$ for some $\hat{N} > 5$, which proves the result of Theorem 6. Our proof of Theorem 6 relies on this new method of incrementally using lower moment bounds to gain control over slightly higher moment bounds, as compared to directly bounding all arbitrarily large moments by some fixed small moment.

Kunze [7] proves this result in his paper with the assumption of initial compact support in the momentum variable. This method allows him to save an entire power of p_0 and use a Gronwall-type inequality to bound the momentum support at time T . We do not have this extra control given by the momentum support of f and needed a wider range of bounds on the K_T term. Actually, our method for bounding K_T can give us strictly better bounds than those of Kunze [7]. In [7], for $2 \leq r < 6$, Kunze proves the bound:

$$\|K_T\|_{L_t^r([0,T];L_x^r)} \leq C_T \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)}. \quad (25)$$

In comparison, we prove the bound in (38) and Proposition 25 which lowers the Lebesgue exponent of the norm on σ_{-1} :

$$\|K_T\|_{L_t^\infty L_x^{mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^q}$$

for $1 \leq m \leq 3$, $q > 3 - \frac{3}{m}$ and $\frac{3m-1}{2m} \leq q \leq \infty$. For the purposes of this problem, obtaining lower Lebesgue exponents yields better estimates because by interpolation

$$\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^q)} \lesssim \|p_0^{2q-1+\nu} f\|_{L_{x,p}^1}$$

for some $\nu > 0$. Thus, lower powers of q yield lower powers of p_0 on the right hand side. This allows us to bound Lebesgue norms of K by lower moments (i.e. lower powers of p_0), which gives us better control on K . We use this extra control on $\|K_T\|_{L_x^r}$ in the range $2 \leq r \leq 6$ by lower moments to help us in the case of initial data with compact support, in which we prove the following:

Theorem 7. *Consider initial data (f_0, E_0, B_0) satisfying the conditions in Theorem 2 and (f, E, B) is the unique classical solution to (1)-(3) in the interval $[0, T)$. Suppose we impose the additional assumption that*

$$\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1 \quad (26)$$

for some $1 \leq r \leq 2$ and some $\beta > 0$ arbitrarily small. Then, we can continuously extend our solution (f, E, B) to an interval $[0, T + \epsilon]$ in C^1 for some $\epsilon > 0$.

The exponent of p_0 in (26) is strictly better than the exponent found in the result stated in Theorem 4 since $\frac{18}{5r} - 1 + \beta < \frac{4}{r} - 1 + \beta$. For example, if $r = 1$, our criteria is $\|p_0^{\frac{13}{5}+\beta} f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1$ which is better than the known criteria of $\|p_0^{3+\beta} f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1$ due to [7]. Similarly for the $r = 2$ case, our criteria is $\|p_0^{\frac{4}{5}+\beta} f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1$ which is better than the known criteria of $\|p_0 f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1$ due to [7].

1.4.2 Outline of Proof

The first step to proving Theorem 7 is to utilize the decomposition

$$|E| + |B| \leq |E_0| + |B_0| + |E_T| + |B_T| + |E_{S,1}| + |B_{S,1}| + |E_{S,2}| + |B_{S,2}| \quad (27)$$

as in Luk-Strain [8]. The advantage to this decomposition is that it allows us to utilize the conservation of the L^2 norm of $|K_g| = (|E \cdot \omega|^2 + |B \cdot \omega|^2 + |E - \omega \times B|^2 + |B + \omega \times E|^2)^{\frac{1}{2}}$ on the space-time cone and it also reduces the power of $1 + \hat{p} \cdot \omega$ in the $K_{S,1}$ term by a power of $\frac{1}{2}$. This decomposition of K_S into the two terms $K_{S,1}$ and $K_{S,2}$ allows us to gain better bounds on the K_S part of the field decomposition. We can bound each element of this decomposition as follows:

$$|K_T| = |E_T| + |B_T| \lesssim W_2(\sigma_{-1})$$

$$|K_{S,1}| = |E_{S,1}| + |B_{S,1}| \lesssim \square^{-1}(|K|\Phi_{-1})$$

$$|K_{S,2}| = |E_{S,2}| + |B_{S,2}| \lesssim (W_2(\sigma_{-1}^2))^{\frac{1}{2}}$$

where

$$\Phi_{-1}(t, x) \stackrel{\text{def}}{=} \max_{|\omega|=1} \int_{\mathbb{R}^3} \frac{f(t, x, p) dp}{p_0(1 + \hat{p} \cdot \omega)^{\frac{1}{2}}}.$$

As in the proof of Theorem 6, we can apply averaging operator estimates to the K_T and $K_{S,2}$ to get the bounds

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{2q}}$$

and

$$\|K_T\|_{L_t^\infty L_x^{mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^q}$$

where $q > 3 - \frac{3}{m}$ and $\frac{3m-1}{2m} \leq q \leq \infty$ for $1 \leq m \leq 3$. (Note that this is where we will use the improved estimate on $\|K_T\|_{L_x^r}$ in the range $2 \leq r \leq 6$, which also give us bounds on the $K_{S,2}$ term. Specifically, in this paper, we use the exponent $r = 4 + \delta$ for some $\delta > 0$ appropriately small.) Using these estimates and using Strichartz estimates, we apply similar techniques as in the proof of Theorem 6 to the $K_{S,1}$ term to obtain bounds on K . We are given better control of the $K_{S,1}$ term because in the inequality $|K_{S,1}| \lesssim \square^{-1}(|K|\Phi_{-1})$, Φ_{-1} has a lower power of singularity in the denominator than σ_{-1} . (We partition our time interval $[0, T]$ under the assumption that $\|\Phi_{-1}\|_{L_t^\infty([0, T]; L_x^2)} \lesssim 1$, which is a weaker assumption than $\|\sigma_{-1}\|_{L_t^\infty([0, T]; L_x^2)} \lesssim 1$.)

The goal of our bound on K in this proof is not to gain bounds on higher moments, as in the proof of Theorem 6. Instead, we use an idea of Pallard [11] and bound the integral of the electric field over the characteristics by appropriate Lebesgue norms involving f and K :

$$|P(T)| \lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} + \|\sigma_{-1}|K| \ln^{\frac{1}{3}}(1 + P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0, T] \times \mathbb{R}^3)}.$$

Using the bounds on $\|K\|_{L_t^\infty([0, T]; L_x^r)}$ for some exponent $r > 4$ appropriately close to 4 and interpolation inequalities, we can then bound these terms by powers of $P(T)$ smaller than 1 to obtain an inequality of the form:

$$P(T) \lesssim 1 + P(T)^\gamma \ln^\lambda(P(t))$$

for some $\gamma \in [0, 1)$ and $\lambda > 0$. From here, we conclude that $P(T) \lesssim 1$.

Our final result improves the continuation criteria due to Luk-Strain in [8]. First, consider a family of planes $\{Q(t)\}_{t \in [0, T]}$. At $t = 0$, we choose a normal vector $n_3(0)$ orthogonal to the plane $Q(0)$ at the origin.

Definition 8. A family of planes $\{Q(t)\}_{t \in [0, T]}$ containing the origin is considered to be **uniformly continuous family of planes** in the following sense: There exists a partition $[T_i, T_{i+1})$ of $[0, T)$ such that locally in a small time interval, for say $s \in [T_i, T_{i+1})$, we can let $n_3(s)$ be the normal to $Q(s)$ at the origin that is on the same half of \mathbb{R}^3 as a $n_3(T_i)$, meaning $\angle(n_3(s), n_3(T_i)) < \angle(n_3(s), -n_3(T_i))$, where $\angle(v, w) \stackrel{\text{def}}{=} \cos^{-1}(\frac{v \cdot w}{|v||w|})$. Then, the map $n_3 : [0, T) \rightarrow \mathbb{S}^2$ is uniformly continuous.

Using this definition, we prove the following:

Theorem 9. Suppose $f_0(x, p) \in H^5(\mathbb{R}^3 \times \mathbb{R}^3)$ with compact support in (x, p) , $E_0, B_0 \in H^5(\mathbb{R}^3)$. Let (f, E, B) be the classical solution in $L_t^\infty([0, T]; H_{x,p}^5) \times L_t^\infty([0, T]; H_x^5) \times L_t^\infty([0, T]; H_x^5)$ to the Vlasov-Maxwell system in $[0, T)$. Let $\{Q(t)\}$ be a uniformly continuous family of planes containing the origin such that there exists a bounded, continuous function $\kappa : [0, T) \rightarrow \mathbb{R}_+$ such that

$$f(t, x, p) = 0 \text{ for } |\mathbb{P}_{Q(t)} p| \geq \kappa(t) \quad \forall x \in \mathbb{R}^3$$

Then there exists $\epsilon > 0$ such that our solution can be extended continuously in time in H^5 to $[0, T + \epsilon]$.

A more general theorem can be proven. Theorem 9 will be a special case of this theorem. First, we need to define a time dependent coordinate system on \mathbb{R}^3 which will depend on the plane $Q(t)$. Let $\{n_1(t), n_2(t), n_3(t)\}$ be unit vectors such that $\{n_1(t), n_2(t)\}$ span $Q(t)$ and $n_3(t)$ is the unit normal to $Q(t)$ as defined earlier.

Fix a time $t \in [0, T]$. By uniform continuity of $n_3(t)$, there exists a partition of $[0, t] = \cup_{i=0}^{n_t} [T_i, T_{i+1})$ (the number of intervals in the partition n_t depends on t and $T_{n_t+1} = t$) such that for $s \in [T_i, T_{i+1})$, we have:

$$\angle(n_3(s), n_3(T_i)) < \angle(-n_3(s), n_3(T_i)) \quad (28)$$

and

$$\angle(n_3(s), n_3(T_i)) < \frac{P(t)^{-1}}{4} \quad (29)$$

We will use this precise partition for the proof of Theorem 10 in this paper.

Theorem 10. *Suppose $f_0(x, p) \in H^5(\mathbb{R}^3 \times \mathbb{R}^3)$ with compact support in (x, p) , $E_0, B_0 \in H^5(\mathbb{R}^3)$. Let (f, E, B) be the classical solution in $L_t^\infty([0, T]; H_{x,p}^5) \times L_t^\infty([0, T]; H_x^5) \times L_t^\infty([0, T]; H_x^5)$ to the Vlasov-Maxwell system in $[0, T]$. Let $\{Q(t)\}$ be a uniformly continuous family of planes containing the origin. Suppose for each $t \in [0, T]$, there exists a measurable, positive function $\kappa : [0, T] \times [0, 2\pi] \rightarrow \mathbb{R}_+$ such that $\kappa(t, \gamma) > 1$,*

$$\sup\{|\mathbb{P}_{Q(t)}p| : \frac{p \cdot n_2(t)}{p \cdot n_1(t)} = \tan(\gamma), f(t, x, p) \neq 0 \text{ for some } x \in \mathbb{R}^3\} < \kappa(t, \gamma)$$

and

$$\int_0^T \left(A(t)^2 + \left(\int_0^t A(s)^8 ds \right)^{\frac{1}{2}} \right) dt < +\infty \text{ where } A(t) = \|\kappa(t, \cdot)\|_{L_\gamma^4}$$

Then there exists $\epsilon > 0$ such that our solution can be extended continuously in time to $[0, T + \epsilon]$.

Note that γ depends on $p \in \mathbb{R}^3$, so we actually have $\tan(\gamma) = \tan(\gamma(p)) = \frac{p \cdot n_2(t)}{p \cdot n_1(t)}$.

1.4.3 Outline of Proof

We modify methods used in [8] to prove Theorem 10. We wish to show that the quantity

$$P(t) = 2 + \sup\{|p| : f(s, x, p) \neq 0 \text{ for some } 0 \leq s \leq t \text{ and } x \in \mathbb{R}^3\}$$

is bounded on $[0, T]$. By the method of characteristics (see [8]), we have the bound

$$P(t) \lesssim 1 + \sup_{(t,x,p) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3} \int_0^t |E(s; X(s; t, x, p))| + |B(s; X(s; t, x, p))| ds$$

We wish to bound the momentum support quantity $P(t)$. To do so, we first find appropriate estimates on E and B . We again use the decomposition:

$$4\pi E(x, t) = (E)_0 + E_{S,1} + E_{S,2} + E_T$$

$$4\pi B(x, t) = (B)_0 + B_{S,1} + E_{S,2} + B_T$$

where $(E)_0$ and $(B)_0$ depend only on the initial data. We have the following estimates from Proposition 3.1 and Proposition 3.4 in [8]:

$$|E_T(t, x)| + |B_T(t, x)| \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{(t-s)^2 p_0^2 (1 + \hat{p} \cdot \omega)^{\frac{3}{2}}} dp \, d\omega \quad (30)$$

$$|E_{S,1}(t, x)| + |B_{S,1}(t, x)| \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} |B| \frac{f(s, x + (t-s)\omega, p)}{(t-s)p_0(1 + \hat{p} \cdot \omega)^{\frac{3}{2}}} dp \, d\omega \quad (31)$$

$$|E_{S,2}(t, x)| + |B_{S,2}(t, x)| \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{(|E \cdot \omega| + |B \cdot \omega| + |B + \omega \times E|) f(s, x + (t-s)\omega, p)}{(t-s)p_0(1 + \hat{p} \cdot \omega)} dp \, d\omega \quad (32)$$

Next, we prove analogous bounds on the momentum integral $\int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{(t-s)^2 p_0^2 (1 + \hat{p} \cdot \omega)^{\frac{3}{2}}} dp$ as those found in [8]. Partitioning the time interval $[0, T]$ into subintervals $[T_i, T_{i+1}]$ small enough as described in (28)

and (29). Applying these two conditions to bound (30), (31) and (32) on each subinterval $[T_i, T_{i+1}]$ in an analogous method to [8]. (These conditions allow us to approximate the integrals in each time subinterval by the integral at the endpoints, as the variation in the momentum support plane is very small on each subinterval. This observation is the key to proving Theorem 10.) We then sum over the partition to prove analogous bounds on the field terms to those found in [8]. From here, we conclude that $P(T) \lesssim 1$ by the bootstrap argument of [8].

2 The operator W_α

The average value of a function $g : X \rightarrow \mathbb{R}$ defined on a space X with finite measure $\nu(X)$ is denoted by

$$\oint_X g(x) dx = \frac{1}{\nu(X)} \int_X g(x) dx \quad (33)$$

Define the operator W_α by

$$W_\alpha(h(t, x)) = \int_{|x-y| \leq t} \frac{h(t-|x-y|, y)}{|x-y|^\alpha} dy = \int_0^t s^{2-\alpha} \oint_{\mathbb{S}^2} h(t-s, x+s\omega) d\mu(\omega) ds \quad (34)$$

where $d\mu(\omega)$ is the spherical measure. Then, by (13) we obtain the following:

Proposition 11. *For the electric and magnetic fields, we have the estimate:*

$$|K_T(t, x)| \lesssim W_2(\sigma_{-1})$$

Thus, we wish to obtain estimates for the operator W_α for $\alpha = 2$. We prove an estimate for general α . Consider:

$$(T_{\alpha,s}h)(t, sx) = s^{2-\alpha} \oint_{\mathbb{S}^2} h(t-s, sx+s\omega) d\mu(\omega) \quad (35)$$

A Schwartz function is a C^∞ function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for any pair of multi-indices, α and β , there exists a finite constant $C_{\alpha,\beta}$ satisfying $\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}$. The set of Schwartz functions form a vector space called the Schwartz space, which is dense in the space L^q for $1 \leq q < \infty$. On the Schwartz space, denoted by $\mathcal{S}(\mathbb{R}^n)$, we have known estimates for the averaging operator $Af = \oint_{\mathbb{S}^2} f(x+\omega) d\mu(\omega)$ from (2) in [5]:

Theorem 12. *The estimate*

$$\|Af\|_{L^a} \lesssim \|f\|_{L^q}, \quad f \in \mathcal{S}(\mathbb{R}^n) \quad (36)$$

holds if and only if $(\frac{1}{q}, \frac{1}{a})$ in the convex hull of $(0, 0)$, $(1, 1)$, and $(\frac{n}{n+1}, \frac{1}{n+1})$.

For the case $n = 3$, the inequality (36) holds for $(\frac{1}{q}, \frac{1}{a})$ in the convex hull of $\{(0, 0), (1, 1), (\frac{3}{4}, \frac{1}{4})\}$. Thus, setting $a = mq$ for $1 \leq m \leq 3$ and for a range of q to be calculated, we have that:

$$\|(T_{\alpha,s}h)(t, sx)\|_{L_x^{mq}} \lesssim s^{2-\alpha} \|h(t-s, sx)\|_{L_x^q} \quad (37)$$

After a change of variables in the spatial coordinates,

$$\|T_{\alpha,s}h(t)\|_{L_x^{mq}} \lesssim s^{2-\alpha+\frac{3}{mq}-\frac{3}{q}} \|h(t-s)\|_{L_x^q}$$

Applying this estimate to the operator W_α under the additional assumption $2 - \alpha + \frac{3}{mq} - \frac{3}{q} > -1$,

$$\begin{aligned} \|W_\alpha h(t)\|_{L_t^r L_x^{mq}} &\leq \left\| \int_0^t \|T_{\alpha,s}h(t)\|_{L_x^{mq}} ds \right\|_{L_t^r} \\ &\lesssim \left\| \int_0^t s^{2-\alpha+\frac{3}{mq}-\frac{3}{q}} \|h(t-s)\|_{L_x^q} ds \right\|_{L_t^r} \\ &\lesssim \left\| \int_0^t s^{2-\alpha+\frac{3}{mq}-\frac{3}{q}} \|h\|_{L_t^\infty L_x^q} ds \right\|_{L_t^r} \\ &\lesssim \|h\|_{L_t^\infty L_x^q} \end{aligned}$$

where $1 \leq r \leq \infty$, $t \in [0, T]$, $x \in \mathbb{R}^3$, and the implicit constant in the upper bound is a continuous function of T , r and α . It remains to check the range of q for which $(\frac{1}{q}, \frac{1}{mq})$ lies in the convex hull described above.

We observe that the line connecting the points $(x, y) = (1, 1)$ and $(x, y) = (\frac{3}{4}, \frac{1}{4})$ is represented by the equation $y = 3x - 2$. Thus, the line $y = \frac{1}{m}x$ meets the line $y = 3x - 2$ when $x = \frac{2m}{3m-1}$.

Summarizing:

Lemma 13. For $1 \leq r \leq \infty$, $1 \leq m \leq 3$, $2 - \alpha + \frac{3}{mq} - \frac{3}{q} > -1$, $\frac{3m-1}{2m} \leq q \leq \infty$,

$$\|W_\alpha h(t, x)\|_{L_t^r([0, T]; L_x^{mq})} \leq C_{T, \alpha} \|h\|_{L_t^\infty([0, T]; L_x^q)} \quad (38)$$

for some explicitly computable constant $C_{T, \alpha}$ depending only on T and α .

3 Estimates on K_T

We can now apply the above estimates to the K_T term. For the $\alpha = 2$, $m = 3$ case, we need $-\frac{2}{q} > -1$ and $\frac{4}{3} \leq q \leq \infty$. Thus, by Proposition 11 and (38), we obtain:

Proposition 14. For $1 \leq r \leq \infty$ and $q > 2$,

$$\|K_T\|_{L_t^r([0, T]; L_x^{3q})} \leq C_T \|\sigma_{-1}\|_{L_t^\infty([0, T]; L_x^q)} \quad (39)$$

for some explicitly computable constant C_T depending only on T .

To get appropriate bounds on K_T , we need to introduce some important interpolation inequalities.

Lemma 15. Let $1 \leq r, s \leq \infty$ and suppose $\nu > 2\frac{r}{s} - 1$. Then

$$\|\sigma_{-1}\|_{L_x^r} \lesssim \|p_0^\nu f\|_{L_x^s L_p^1}. \quad (40)$$

Proof. By Hölder's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$:

$$\int_{\mathbb{R}^3} \frac{f(t, x, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \leq \left(\int_{\mathbb{R}^3} \frac{dp}{p_0^{(1+\alpha)q'}(1 + \hat{p} \cdot \omega)^{q'}} \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^3} p_0^{\alpha q} f(t, x, p)^q dp \right)^{\frac{1}{q}}$$

Call the first term on the right hand side I . In order to bound this term, we use the standard inequality

$$(1 + \hat{p} \cdot \omega)^{-1} \lesssim \min\{p_0^2, \theta^{-2}\} \quad (41)$$

where $\theta = \angle(\frac{p}{|p|}, -\omega) \in [0, \pi]$. Note that for small θ , we can assume $\sin(\theta) \approx \theta$. Assuming $\alpha > 2 - \frac{1}{q}$:

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \frac{dp}{p_0^{(1+\alpha)q'}(1 + \hat{p} \cdot \omega)^{q'}} \\ &\leq \int_{\mathbb{R}^3} \frac{dp}{p_0^{(\alpha-1)q'+2}(1 + \hat{p} \cdot \omega)} \\ &\lesssim \lim_{P \rightarrow \infty} \int_0^P d|p| \int_0^{2\pi} d\phi \left(\int_0^{p_0^{-1}} \frac{p_0^2 |p|^2 \sin(\theta) d\theta}{p_0^{(\alpha-1)q'+2}} + \int_{p_0^{-1}}^\pi \frac{|p|^2 \sin(\theta) d\theta}{p_0^{(\alpha-1)q'+2} \theta^2} \right) \\ &\lesssim \lim_{P \rightarrow \infty} \int_0^P d|p| \frac{1 + \log(p_0)}{p_0^{(\alpha-1)q'}} \\ &\lesssim 1 \end{aligned} \quad (42)$$

since $(\alpha-1)q' > 1$. Taking L^r norm on both sides and using the conservation law $\|f\|_{L_{x,p}^\infty} \lesssim 1$, we obtain:

$$\|\sigma_{-1}\|_{L_x^r} \lesssim \|p_0^{\alpha q} f^q\|_{L_x^{\frac{r}{q}} L_p^1}^{\frac{1}{q}} \lesssim \|p_0^{\alpha q} f\|_{L_x^{\frac{r}{q}} L_p^1}^{\frac{1}{q}} \quad (43)$$

Hence, setting $q = \frac{r}{s}$, we finally have for $\nu > \frac{2r}{s} - 1$:

$$\|\sigma_{-1}\|_{L_x^r} \lesssim \|p_0^\nu f\|_{L_x^s L_p^1}^{\frac{s}{r}} \quad (44)$$

This completes the proof of this inequality. \square

We have the following interpolation-type inequality from Proposition 10.3 in [9]:

Proposition 16. *Suppose η , ρ , and τ are real numbers such that $0 < q\eta < 1$ and*

$$\tau \geq \frac{\rho - \eta(N + 3 - 3q)}{1 - q\eta}$$

Then,

$$\|fp_0^\rho\|_{L_t^\infty([0,T];L_x^q L_p^1)} \lesssim \|fp_0^\tau\|_{L_t^\infty([0,T];L_x^q L_p^1)}^{1-q\eta} \|fp_0^N\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^\eta \quad (45)$$

Applying Lemma 15 and (45) to (39), we obtain for $N > 3$, $1 \leq r \leq \infty$ and some $\delta > 0$:

$$\begin{aligned} \|K_T\|_{L_t^\infty([0,T];L_x^{N+3})}^{N+3} &\lesssim \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{\frac{N+3}{3}})}^{N+3} \\ &\lesssim \|p_0^{\frac{2N+3+\delta}{3}} f\|_{L_x^1 L_p^1}^3 \\ &\lesssim \|p_0^{\frac{2N-3N\eta+3+\delta}{3(1-\eta)}} f\|_{L_x^1 L_p^1}^{3-3\eta} \|p_0^N f\|_{L_x^1 L_p^1}^{3\eta} \end{aligned}$$

Setting $\eta = \frac{1-\gamma}{3}$, we obtain the needed estimate for K_T :

Proposition 17. *Given $1 \leq r \leq \infty$, $\gamma \in (0, 1)$, $N > 3$ and $\delta > 0$,*

$$\|K_T\|_{L_t^r([0,T];L_x^{N+3})}^{N+3} \lesssim \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^{2+\gamma} \|p_0^N f\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^{1-\gamma} \quad (46)$$

4 Estimates on K_S

Recall Strichartz estimates for the wave operator from (4.7) in Sogge [12]:

Theorem 18. (*Strichartz Estimates*)

Given $\lambda \in (0, 1)$ and solution $u : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ to $\square u = F$ on $[a, b] \times \mathbb{R}^3$ with initial data $u|_{t=a} = u(a)$ and $\partial_t u|_{t=a} = \partial_t u(a)$, there exists a constant C_λ such that

$$\|u\|_{L_t^{\frac{2}{\lambda}}([a,b];L^{\frac{2}{1-\lambda}})} + \|u\|_{L_t^\infty([a,b];\dot{H}^\lambda)} + \|\partial_t u\|_{L_t^\infty([a,b];\dot{H}^\lambda)} \leq C_\lambda (\|F\|_{L_t^{\frac{2}{1+\lambda}}([a,b];L^{\frac{2}{2-\lambda}})} + \|u(a)\|_{\dot{H}^\lambda} + \|\partial_t u(a)\|_{\dot{H}^\lambda})$$

Recall from the preliminary estimates (14) and (15) for K_S that

$$|K_S| \lesssim \square^{-1}(|K|\sigma_{-1}) \quad (47)$$

Using Strichartz estimates, we can prove the following:

Proposition 19. *Assume $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \lesssim 1$. Given $N > 3$, $\gamma \in (0, 1)$ and $\delta > 0$, we obtain the estimate*

$$\|K_S\|_{L_t^1([0,T];L_x^{N+3})}^{N+3} \lesssim 1 + \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^{2+\gamma} \|p_0^N f\|_{L_t^\infty([0,T];L_x^1 L_p^1)}^{1-\gamma} \quad (48)$$

Proof. By the above estimate (47):

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \leq \|\square^{-1}(|K|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \quad (49)$$

Using the decomposition $K = K_0 + K_T + K_S$, we obtain for some time interval $[a, b] \subset [0, T]$:

$$\begin{aligned} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} &\leq \|\square^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} + \\ &\quad \|\square^{-1}(|K_T|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} + \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \end{aligned} \quad (50)$$

Fix an interval $[a, b] \subset [0, T]$. First notice that

$$\|\square^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \lesssim \|\square^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T];L_x^{N+3})}$$

and similarly

$$\|\square^{-1}(|K_T|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \lesssim \|\square^{-1}(|K_T|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T];L_x^{N+3})}$$

Setting $\lambda = \frac{N+1}{N+3}$, we obtain by the Strichartz estimates for the wave operator:

$$\|\square^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T];L_x^{N+3})} \lesssim C_{\frac{N+1}{N+3}} \| |K_0|\sigma_{-1} \|_{L_t^{\frac{(N+3)}{N+2}}([0,T];L_x^{\frac{2(N+3)}{N+5}})}$$

since we have trivial initial data by (15). Applying the same argument to the K_T term and by (50) we obtain:

$$\begin{aligned} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} &\leq C_{\frac{N+1}{N+3}} \| |K_0|\sigma_{-1} \|_{L_t^{\frac{(N+3)}{N+2}}([0,T];L_x^{\frac{2(N+3)}{N+5}})} + C_{\frac{N+1}{N+3}} \| |K_T|\sigma_{-1} \|_{L_t^{\frac{(N+3)}{N+2}}([0,T];L_x^{\frac{2(N+3)}{N+5}})} \\ &\quad + \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \end{aligned} \quad (51)$$

Applying Hölder's inequality with $\frac{1}{2} + \frac{1}{N+3} = \frac{N+5}{2(N+3)}$:

$$\begin{aligned} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} &\leq C_{\frac{N+1}{N+3}} \|K_0\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T];L_x^{N+3})} \|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)} \\ &\quad + C_{\frac{N+1}{N+3}} \|K_T\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T];L_x^{N+3})} \|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)} + \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \end{aligned} \quad (52)$$

Note that we can bound $C_{\frac{N+1}{N+3}} \|K_0\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T];L_x^{N+3})} \|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)}$ by a constant since K_0 depends only on initial data and we have assumed that $\|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)} \lesssim_T \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \lesssim 1$. Finally, using the estimate (46) on K_T from the previous section,

$$\begin{aligned} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} &\leq (data) + C \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f \|p_0^N\|^{\frac{2+\gamma}{N+3}} f \|p_0^N\|^{\frac{1-\gamma}{N+3}} \|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)} \\ &\quad + \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \end{aligned} \quad (53)$$

Similarly, we can now apply Strichartz estimates and Hölder's inequality to the K_S term. Note that we kept the time interval on the K_S term as $[a, b]$. Setting $u = \square^{-1}(|K_S|\sigma_{-1})$,

$$\begin{aligned} \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} &\leq C_{\frac{N+1}{N+3}} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \|\sigma_{-1}\|_{L_t^2([a,b];L_x^2)} \\ &\quad + \|u(a)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(a)\|_{\dot{H}^{\frac{N+1}{N+3}}} \end{aligned} \quad (54)$$

Next, we can choose a partition $0 = T_0 < T_1 < T_2 < \dots < T_{k-1} < T_k = T$ of $[0, T]$ such that

$$\|\sigma_{-1}\|_{L_t^2([T_i, T_{i+1}];L_x^2)} \leq \frac{1}{2C^{\frac{N+1}{N+3}}} \text{ for } i \in \{0, 1, \dots, k-1\}$$

due to the assumption $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \leq \tilde{C}$ for some \tilde{C} . (For example, we can choose our partition so that $(T_i - T_{i-1})^{\frac{1}{2}} \leq \frac{1}{2\tilde{C}2C^{\frac{N+1}{N+3}}}$ for $i = 1, 2, \dots, k$.) Using (53) and (54):

$$\begin{aligned} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i, T_{i+1}];L_x^{N+3})} &\leq 2(data)_i + 2C \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f \|p_0^N\|^{\frac{2+\gamma}{N+3}} f \|p_0^N\|^{\frac{1-\gamma}{N+3}} \|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)} \\ &\quad + 2C_{\frac{N+1}{N+3}} (\|u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}}) \end{aligned} \quad (55)$$

Thus, by Hölder's inequality, our choice of partition, and then (55):

$$\begin{aligned}
& \| |K_S| \sigma_{-1} \|_{L_t^{\frac{N+3}{N+2}}([T_i, T_{i+1}]; L_x^{\frac{2(N+3)}{N+5}})} \leq \frac{1}{2C_{\frac{N+1}{N+3}}} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i, T_{i+1}]; L_x^{N+3})} \\
& \leq \frac{1}{C_{\frac{N+1}{N+3}}} (data)_i + \frac{1}{C_{\frac{N+1}{N+3}}} C \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f \|p_0^N f\|^{\frac{2+\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \|p_0^N f\|^{\frac{1-\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \\
& \quad + \|u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} \quad (56)
\end{aligned}$$

Using the Strichartz estimates again for $[T_{i-1}, T_i]$, we obtain:

$$\begin{aligned}
& \|u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} \\
& \leq C_{\frac{N+1}{N+3}} \left(\|u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \| |K_S| \sigma_{-1} \|_{L_t^{\frac{N+3}{N+2}}([T_{i-1}, T_i]; L_x^{\frac{2(N+3)}{N+5}})} \right) \\
& \leq (data)_i + C \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f \|p_0^N f\|^{\frac{2+\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \|p_0^N f\|^{\frac{1-\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \\
& \quad + 2C_{\frac{N+1}{N+3}} (\|u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}})
\end{aligned}$$

Thus, since $u(0) = \partial_t u(0) = 0$, we do an iteration of the above to get the following estimate:

$$\begin{aligned}
& \|u(T_{j+1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_{j+1})\|_{\dot{H}^{\frac{N+1}{N+3}}} \\
& \leq \sum_{i=0}^j (2C_{\frac{N+1}{N+3}})^{j-i} \left((data)_i + C \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f \|p_0^N f\|^{\frac{2+\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \|p_0^N f\|^{\frac{1-\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \right) \quad (57)
\end{aligned}$$

Plugging this estimate into (55) and using the triangle inequality to sum over the entire partition,

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i, T_{i+1}]; L_x^{N+3})} \lesssim \sum_{i=0}^{k-1} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i, T_{i+1}]; L_x^{N+3})} \lesssim 1 + \|p_0\|^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f \|p_0^N f\|^{\frac{2+\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)} \|p_0^N f\|^{\frac{1-\gamma}{N+3}}_{L_t^\infty([0, T]; L_x^1 L_p^1)}$$

which implies the estimate (48) after an application of Hölder's inequality in the time variable. \square

5 Bounds on Higher Moments

We have the standard moment estimate recalled from Proposition 7.3 in [9]:

Proposition 20. *Given $N > 0$, we have the uniform estimate*

$$\|p_0^N f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)} \lesssim \|p_0^N f_0\|_{L_t^\infty([0, T]; L_x^1 L_p^1)} + \|K\|_{L_t^1([0, T]; L_x^{N+3})}^{N+3}$$

It suffices to bound moments $\|p_0^N f\|_{L_t^\infty([0, T]; L_x^1 L_p^1)}$ for all $N > 0$ due to the following sharpened estimate from [9]:

Proposition 21. *Over any spatial characteristic curve $X(s; t, x, p)$ we have the bound:*

$$\begin{aligned}
& \sup_{(t, x, p) \in \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_p^3} \int_0^{T_*} (|E(s, X(s; t, x, p))| + |B(s, X(s; t, x, p))|) ds \lesssim 1 + \|K \sigma_{-1}\|_{L_t^\infty([0, T]; L_x^{2+\lambda} L_p^1)} \\
& \quad + \|\sigma_{-1}\|_{L_t^\infty([0, T]; L_x^{3+\tilde{\lambda}} L_p^1)} \quad (58)
\end{aligned}$$

for any $\lambda, \tilde{\lambda} > 0$.

Proof. We can rewrite the bounds (13) and (14) in the form:

$$|K_T|(t, x) \lesssim \int_{C_{t, x}} \frac{\sigma_{-1}(s, y)}{(t-s)^2} d\sigma \quad (59)$$

$$|K_S|(t, x) \lesssim \int_{C_{t,x}} \frac{(|K|\sigma_{-1})(s, y)}{t-s} d\sigma \quad (60)$$

where the integral over the cone $C_{t,x}$ is given (5). Using (59) and (60), we can bound our integral over the characteristic $X(s; t, x, p)$ by:

$$\begin{aligned} \sup \int_0^{T_*} (|E(s, X(s; t, x, p))| + |B(s, X(s; t, x, p))|) ds \\ \lesssim 1 + \int_0^{T_*} \int_{C_{s,X(s)}} \frac{\sigma_{-1}(\tilde{s}, y)}{(s-\tilde{s})^2} d\sigma ds + \int_0^{T_*} \int_{C_{s,X(s)}} \frac{(|K|\sigma_{-1})(\tilde{s}, y)}{(s-\tilde{s})} d\sigma ds \end{aligned} \quad (61)$$

where $d\sigma = d\sigma(\tilde{s}, y) = (s-\tilde{s})^2 \sin(\theta) d\tilde{s} d\phi d\theta$ and $X(s) = X(s; t, x, p)$. The integral terms on the right hand side have the general form:

$$I_i(t, x) \stackrel{\text{def}}{=} \int_0^t \int_{C_{s,X(s)}} \frac{g_i}{(s-\tilde{s})^i} d\sigma ds \quad (i = 1, 2), \quad (62)$$

where $g_1 = |K|\sigma_{-1}$ and $g_2 = \sigma_{-1}$. By a change of variables after writing (62) expanded as (5), we obtain:

$$I_i(t, x) = \int_0^t \int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi (s-\tilde{s})^{2-i} \sin(\theta) g_i(s, X(s) + (s-\tilde{s})\omega) d\theta d\phi ds d\tilde{s} \stackrel{\text{def}}{=} \int_0^t J_i(\tilde{s}, X(\tilde{s})) d\tilde{s}, \quad (63)$$

where again we have adopted the convention $X(s) = X(s; t, x, p)$.

Following Pallard [10], we define the diffeomorphism $\pi \stackrel{\text{def}}{=} X(s) + (s-\tilde{s})\omega$. This change of variables has Jacobian $J_\pi = (X'(s) \cdot \omega + 1)(s-\tilde{s})^2 \sin(\theta) \neq 0$ on $\theta \in (0, \pi)$ (since $|X'(s)| \leq |\hat{V}(s)| < 1$ and hence $X'(s) \cdot \omega + 1 > 0$). First using Hölder's inequality for Hölder exponents q, q' such that $\frac{1}{q} + \frac{1}{q'} = 1$:

$$\begin{aligned} J_i(\tilde{s}, X(\tilde{s})) \leq \left(\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_\pi^{\frac{q'}{q}}} d\theta d\phi ds \right)^{\frac{1}{q'}} \\ \times \left(\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi g_i(s, X(s) + (s-\tilde{s})\omega)^q J_\pi d\theta d\phi ds \right)^{\frac{1}{q}} \end{aligned} \quad (64)$$

Next, using the change of variables described by the diffeomorphism π in the second integral on the right hand side of (64):

$$J_i(\tilde{s}, X(\tilde{s})) \leq \left(\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_\pi^{\frac{q'}{q}}} d\theta d\phi ds \right)^{\frac{1}{q'}} \|g_i(\tilde{s})\|_{L^q(\mathbb{R}^3)} \quad (65)$$

Finally, plugging in the expression for J_π into the remaining integral on the right hand side, we see that it is bounded for certain choices of q . To see this, choose coordinates (θ, ϕ) such that $X' \cdot \omega = -|X'| |\omega| \cos(\theta) \geq -\cos(\theta)$. Then, using this coordinate system:

$$\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_\pi^{\frac{q'}{q}}} d\theta d\phi ds \lesssim \int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi \frac{(s-\tilde{s})^{(2-i)q' - \frac{2q'}{q}} \sin^{q' - \frac{q'}{q}}(\theta)}{(1 - \cos(\theta))^{\frac{q'}{q}}} d\theta d\phi ds \quad (66)$$

Now, note that $\frac{1}{1 - \cos(\theta)} = \frac{1}{1 - \sqrt{1 - \sin^2(\theta)}} = \frac{1 + \sqrt{1 - \sin^2(\theta)}}{\sin^2(\theta)} \lesssim \frac{1}{\sin^2(\theta)}$ since $1 + \sqrt{1 - \sin^2(\theta)} \leq 2$. Plugging this into the above, we obtain:

$$\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_\pi^{\frac{q'}{q}}} d\theta d\phi ds = \int_{\tilde{s}}^t \int_0^{2\pi} \int_0^\pi (s-\tilde{s})^{(2-i)q' - \frac{2q'}{q}} \sin^{q' - \frac{3q'}{q}}(\theta) d\theta d\phi ds \quad (67)$$

The integral over θ remains bounded when $q' - \frac{3q'}{q} > -1$ and $(2-i)q' - \frac{2q'}{q} > -1$, i.e. when $q > 2$ and $q > \frac{3}{3-i}$ for $i = 1, 2$. \square

From the above estimate, we can use Hölder's inequality and Lemma 15 to obtain for any $\lambda', \tilde{\lambda}, \hat{\lambda} > 0$:

$$\begin{aligned} & \|K\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{2+\lambda}L_p^1)} + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{3+\tilde{\lambda}}L_p^1)} \\ & \lesssim \|K\|_{L_t^\infty([0,T];L_x^{3+\lambda'})} \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^3L_p^1)} + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{3+\tilde{\lambda}}L_p^1)} \\ & \lesssim \|K\|_{L_t^\infty([0,T];L_x^{3+\lambda'})} \|p_0^{5+2\hat{\lambda}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1}{3+\tilde{\lambda}}} + \|p_0^{5+2\tilde{\lambda}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1}{3+\tilde{\lambda}}} \end{aligned} \quad (68)$$

By interpolation and the conservation law (9), there exists some $\theta \in (0, 1)$ such that

$$\|K\|_{L_t^\infty([0,T];L_x^{3+\lambda'})} \lesssim \|K\|_{L_t^\infty([0,T];L_x^{6+\lambda'})}^\theta \|K\|_{L_t^\infty([0,T];L_x^2)}^{1-\theta} \lesssim \|K\|_{L_t^\infty([0,T];L_x^{6+\lambda'})}^\theta.$$

By the estimates (46) and (48), we can further bound this by:

$$\|K\|_{L_t^\infty([0,T];L_x^{6+\lambda'})} \lesssim 1 + \|p_0^{\frac{(3+\lambda')(1+\gamma)+3+\delta}{2+\gamma}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{2+\gamma} \|p_0^{3+\lambda'}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{1-\gamma} \quad (69)$$

for any $\delta > 0$ and $\gamma \in (0, 1)$. Choosing $\delta < \lambda'$, we obtain that

$$\frac{(3+\lambda')(1+\gamma)+3+\delta}{2+\gamma} < 3+\lambda',$$

and hence by (69):

$$\|K\|_{L_t^\infty([0,T];L_x^{6+\lambda'})} \lesssim 1 + \|p_0^{3+\lambda'}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^3.$$

Putting these bounds together, we obtain:

$$\begin{aligned} & \|K\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{2+\lambda}L_p^1)} + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{3+\tilde{\lambda}}L_p^1)} \lesssim (1 + \|p_0^{3+\lambda'}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^3) \|p_0^{5+2\hat{\lambda}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1}{3+\tilde{\lambda}}} \\ & \quad + \|p_0^{5+2\tilde{\lambda}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1}{3+\tilde{\lambda}}} \end{aligned} \quad (70)$$

Thus, in order to satisfy the known continuation criteria stated in Theorem 3, we simply need to bound $\|p_0^{5+\lambda}f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1$ for some $\lambda > 0$. To this end, we can use the estimates on K to prove that:

Proposition 22. *Consider initial data f_0 such that $\|p_0^N f_0\|_{L_x^1L_p^1} \lesssim 1$ and suppose we have the bound $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} + \|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1$ where $M > \frac{N+3}{2}$ for some $N > 3$. Then*

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1$$

and

$$\|K\|_{L_t^\infty([0,T];L_x^{N+3})} \lesssim 1.$$

Proof. By the estimates on K_T and K_S given by (46) and (48) respectively and Proposition 20, we obtain for some $\gamma \in (0, 1)$:

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1 + \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{2+\gamma} \|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{1-\gamma} \quad (71)$$

Choose appropriate $0 < \gamma < 1$ and $\delta > 0$ such that $\frac{N(1+\gamma)+3+\delta}{2+\gamma} = M$ and let the implicit constant in (71) be denoted by $C > 0$. (Suppose $M = \frac{N+3+\epsilon}{2}$. Then set $\delta = \epsilon + \left(\frac{N+3+\epsilon}{2} - N\right)\gamma$. For $\gamma \in (0, 1)$ sufficiently small, $\delta > 0$.) Thus, since $\|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \leq B$ for some constant $B > 0$ and by Young's inequality:

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \leq C + CB^\beta \|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{1-\gamma} \leq C + \gamma C^{\frac{1}{\gamma}} B^{\frac{2+\gamma}{\gamma}} + (1-\gamma) \|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \quad (72)$$

Thus for some constant \tilde{C} ,

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \leq \frac{1}{\gamma} (C + \gamma \tilde{C} B^{\frac{2+\gamma}{\gamma}}) \lesssim 1 \quad (73)$$

Finally, plugging (73) into (46) and (48), we obtain that $\|K\|_{L_t^\infty([0,T];L_x^{N+3})} \lesssim 1$. \square

Theorem 23. Suppose $\|p_0^{\tilde{N}} f_0\|_{L_x^1 L_p^1} \lesssim 1$ for some $\tilde{N} > 5$. Let $M > 3$. Then $\|p_0^M f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1$ is a continuation criteria for the Vlasov-Maxwell system without compact support.

Proof. First, if $M > 5$, then by the comment under (70), we are done. (Note that this is also a known continuation criteria found in [9].) If $3 < M < 5$, note that by Lemma 15

$$\|\sigma_{-1}\|_{L_t^\infty([0,T]; L_x^2)} + \|p_0^M f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim \|p_0^M f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1.$$

Suppose $M > 3$ and $\|p_0^M f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1$. Since $\|p_0^{\tilde{N}} f_0\|_{L_x^1 L_p^1} \lesssim 1$ for some $\tilde{N} > 5$, it follows that $\|p_0^N f_0\|_{L_x^1 L_p^1} \lesssim 1$ for all $N < 5$. Then, by Proposition 22, we obtain that $\|p_0^N f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1$ for $N = 2M - 3 - \delta$ for $\delta > 0$ as long as $3 < 2M - 3 - \delta < 5$. Note that if $M > 3$, then $2M - 3 = M + M - 3 > M$. Hence setting $\delta = \frac{M-3}{2}$, we obtain that $N = 2M - 3 - \delta = 2M - 3 - \frac{M-3}{2} = M + \frac{M-3}{2} > M > 3$.

Let $M = M_0$ and suppose, as above, that

$$\|p_0^M f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} = \|p_0^{M_0} f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1.$$

Then, if $M_1 = M_0 + \frac{M_0-3}{2} < 5$, we know by the above that

$$\|p_0^{M_1} f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1.$$

Define the sequence M_i in this manner: let $M_{i+1} = M_i + \frac{M_i-3}{2}$. Notice that since $M_0 > 3$, by the earlier argument, we obtain that $M_1 > 3$. By induction, we obtain that $M_k > 3$ for all $k \in \mathbb{N}$.

Since $M = M_0 > 3$, there exists an $\epsilon > 0$ such that $M_0 = 3 + \epsilon$. We now claim that $M_k > M_0 + \frac{k\epsilon}{2}$. Indeed, this is true in the case of M_0 . Suppose it holds for $k = n$. Then, since $M_n - 3 > M_0 - 3 + \frac{n\epsilon}{2} = \frac{\epsilon}{2} + \frac{n\epsilon}{4} > \frac{\epsilon}{2}$, it follows that $M_{n+1} = M_n + \frac{M_n-3}{2} > M_0 + \frac{n\epsilon}{2} + \frac{\epsilon}{2} = M_0 + \frac{(n+1)\epsilon}{2}$.

Thus, as n tends to infinity, we know that M_n tends to infinity. Thus, there exists some $m \in \mathbb{N}$ such that $3 < M_m < 5$ but $M_{m+1} > 5$. Under our assumption that $\|p_0^{M_0} f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1$, we can iterate the argument above to obtain that $\|p_0^{M_n} f\|_{L_t^\infty([0,T]; L_x^1 L_p^1)} \lesssim 1$ for all positive integers $n \leq m$. Finally, choose some $\delta > 0$ such that $5 < 2M_m - 3 - \delta < \tilde{N}$. (This is certainly possible since choosing $\delta = \frac{M_m-3}{2}$, we obtain by our choice of m that $M_{m+1} = 2M_m - 3 - \delta > 5$. On the other hand, if $M_{m+1} > \tilde{N} > 5$, we simply choose a large delta such that $2M - 3 - \delta$ is still greater than 5 but is less than \tilde{N} .) Let us set $\tilde{M} = 2M_m - 3 - \delta$. Since $\tilde{M} < \tilde{N}$, we know that $\|p_0^{\tilde{M}} f_0\|_{L_x^1 L_p^1} \lesssim 1$. By Proposition 22, we obtain that $\|p_0^{\tilde{M}} f\|_{L_x^1 L_p^1} \lesssim 1$. Since $\tilde{M} > 5$, by the comment under (70), we are done. \square

6 Another Field Decomposition

In this section, we recall the decomposition found in Luk-Strain [8] and bound each piece in the form of the operator W_2 or the inverse d-Alembertain \square^{-1} . Note that from this point in this paper, we define

$$|K| \stackrel{\text{def}}{=} |E| + |B|$$

Then, we have that $|K| \leq |K_0| + |K_T| + |K_{S,1}| + |K_{S,2}|$ where

Proposition 24. We have the following estimates:

$$|K_T| = |E_T| + |B_T| \lesssim W_2(\sigma_{-1}) \tag{74}$$

$$|K_{S,1}| = |E_{S,1}| + |B_{S,1}| \lesssim \square^{-1}(|K|\Phi_{-1}) \tag{75}$$

$$|K_{S,2}| = |E_{S,2}| + |B_{S,2}| \lesssim (W_2(\sigma_{-1}^2))^{\frac{1}{2}} \tag{76}$$

where

$$\Phi_{-1}(t, x) \stackrel{\text{def}}{=} \max_{|\omega|=1} \int_{\mathbb{R}^3} \frac{f(t, x, p) dp}{p_0(1 + \hat{p} \cdot \omega)^{\frac{1}{2}}}. \tag{77}$$

Proof. Following the decomposition of [8] we have that

$$(|E_T| + |B_T|)(t, x) \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{(t-s)^2 p_0(1 + \hat{p} \cdot \omega)} dp d\sigma(\omega) \quad (78)$$

Using the change of variable $t-s \rightarrow s$ and writing the integral over the cone $C_{t,x}$ as an integral over spheres of radius s , we obtain:

$$(|E_T| + |B_T|)(t, x) \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(t-s, x + s\omega, p)}{s^2 p_0(1 + \hat{p} \cdot \omega)} dp d\sigma(\omega) \leq \int_0^t \int_{\mathbb{S}^2} \sigma_{-1}(t-s, x + s\omega) d\sigma(\omega) \quad (79)$$

which is of the form $W_2(\sigma_{-1})$.

Next, by Proposition 3.4 in [8]:

$$(|E_{S,1}| + |B_{S,1}|)(t, x) \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{|B|f(s, x + (t-s)\omega, p)}{(t-s)p_0(1 + \hat{p} \cdot \omega)^{\frac{1}{2}}} dp d\sigma(\omega) \lesssim \int_{C_{t,x}} \frac{|B|\Phi_{-1}(s, x + (t-s)\omega)}{t-s} d\sigma(\omega) \quad (80)$$

But since $|B| \leq |K|$, we finally obtain:

$$(|E_{S,1}| + |B_{S,1}|)(t, x) \lesssim \int_{C_{t,x}} \frac{|K|\Phi_{-1}(s, x + (t-s)\omega)}{t-s} d\sigma(\omega) \quad (81)$$

which is (76). Recall that this is precisely the representation formula for the inhomogeneous wave equation of the form:

$$\square u = |K|\Phi_{-1}; \quad u|_{t=0} = \partial_t u|_{t=0} = 0$$

Finally, our last term has the following bound from Proposition 3.4 in [8]:

$$(|E_{S,2}| + |B_{S,2}|)(t, x) \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{|K_g|f(s, x + (t-s)\omega)}{(t-s)p_0(1 + \hat{p} \cdot \omega)} dp d\sigma(\omega) \quad (82)$$

where $|K_g|^2 = |E \cdot \omega|^2 + |B \cdot \omega|^2 + |E - \omega \times B|^2 + |B + \omega \times E|^2$. Recall the conservation law $\|K_g\|_{L^2(C_{t,x})} \lesssim 1$ from Proposition 2.2 in [8] and use Hölder's inequality to obtain:

$$(|E_{S,2}| + |B_{S,2}|)(t, x) \lesssim \left(\int_{C_{t,x}} \left(\int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega)}{(t-s)p_0(1 + \hat{p} \cdot \omega)} dp \right)^2 d\sigma(\omega) \right)^{\frac{1}{2}} \quad (83)$$

Finally, using the same change of variables as in (79), we get (76). \square

7 New Bounds on $|K|$

From this point in the paper, we adopt the convention that $\rho+$ denotes some appropriate number $\rho + \epsilon$ where $\epsilon > 0$ is very small, $\epsilon \ll 1$. Note that the size of ϵ may vary depending on the term, but the key point is that ϵ is appropriately small in each of the estimates below. Similarly, we let $\rho-$ denote some appropriate number $\rho - \epsilon$ for $\epsilon \ll 1$ chosen to be appropriately small.

Proposition 25. *Given $1 \leq m \leq 3$, $\frac{3}{mq} - \frac{3}{q} > -1$ and $\frac{3m-1}{2m} \leq q \leq \infty$, we have the estimate:*

$$\|K_T\|_{L_t^\infty L_x^{mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^q} \quad (84)$$

Proof. By (74), we can apply (38) for $\alpha = 2$ to $|K_T|$. \square

In particular if $m = 2$, then we need $-\frac{3}{2q} > -1$ (or $q > \frac{3}{2}$) and $q \geq \frac{5}{4}$. Hence, we have for $q > 2$:

$$\|K_T\|_{L_t^\infty L_x^{4+}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{2+}} \quad (85)$$

Proposition 26. *Given $1 \leq m \leq 3$, $\frac{3}{mq} - \frac{3}{q} > -1$ and $\frac{3m-1}{2m} \leq q \leq \infty$,*

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{2q}} \quad (86)$$

Proof. By (76):

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|W_2((\sigma_{-1})^2)^{\frac{1}{2}}\|_{L_t^\infty L_x^{2mq}} = \|W_2((\sigma_{-1})^2)\|_{L_t^\infty L_x^{mq}}^{\frac{1}{2}} \quad (87)$$

We can apply (38) now to get:

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|(\sigma_{-1})^2\|_{L_t^\infty L_x^q}^{\frac{1}{2}} = \|\sigma_{-1}\|_{L_t^\infty L_x^{2q}} \quad (88)$$

□

For reasons that will be clear in Section 9, we use Proposition 25 and Proposition 26 to bound the quantities $\|K_T\|_{L_t^\infty L_x^{4+}}$ and $\|K_{S,2}\|_{L_t^\infty L_x^{4+}}$. Using Proposition 26, we can compute for $|K_{S,2}|$:

$$\|K_{S,2}\|_{L_t^\infty L_x^{4+}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \quad (89)$$

where we used $m = \frac{5}{3}$ and $q = \frac{6}{5}+$. In particular, setting $q = \frac{6+\epsilon}{5}$ for $\epsilon \ll 1$, we see that m and q satisfy the conditions of Proposition 26. The explicit estimate written in (89) is

$$\|K_{S,2}\|_{L_t^\infty L_x^{4+\frac{2\epsilon}{3}}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+\frac{2\epsilon}{5}}}.$$

Similarly, by Proposition 25, we can compute for $|K_T|$:

$$\|K_T\|_{L_t^\infty L_x^{4+}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \quad (90)$$

where we used $m = \frac{5}{3}$ and $q = \frac{12}{5}+$. Note that this is not the lowest Lebesgue norm exponent that can be chosen for σ_{-1} . However, we do not have a better bound in the $K_{S,2}$ estimate, so a better estimate on the K_T term is not useful.

Finally, we employ an iteration argument using Strichartz estimates for the inhomogeneous wave equation to gain bounds on $K_{S,1}$. For the remainder of the paper, assume that

$$\|\Phi_{-1}\|_{L_t^\infty L_x^2} \lesssim 1 \quad (91)$$

Proposition 27. *We have the following bound on $K_{S,1}$ assuming (91):*

$$\|K_{S,1}\|_{L_t^\infty([0,T];L_x^{4+})} \lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{\frac{12}{5}+})} \quad (92)$$

Proof. For $\gamma \in (0, 1)$, we obtain by (75) for some interval $[a, b] \subset [0, T]$:

$$\|K_{S,1}\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([a,b] \times \mathbb{R}^3)} \lesssim \|\square^{-1}(|K|\Phi_{-1})\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([a,b] \times \mathbb{R}^3)} \quad (93)$$

(Note that we will set $\gamma = \frac{1}{2}+$ later in the proof.) Applying the triangle inequality to the decomposition $|K| \leq |K_0| + |K_T| + |K_{S,1}| + |K_{S,2}|$ and extending the interval $[a, b]$ to $[0, T]$ on certain terms, we obtain:

$$\begin{aligned} & \|K_{S,1}\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([a,b] \times \mathbb{R}^3)} \\ & \leq \|\square^{-1}(|K_0|\Phi_{-1})\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([0,T] \times \mathbb{R}^3)} + \|\square^{-1}(|K_T|\Phi_{-1})\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([0,T] \times \mathbb{R}^3)} \\ & \quad + \|\square^{-1}(|K_{S,2}|\Phi_{-1})\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([0,T] \times \mathbb{R}^3)} + \|\square^{-1}(|K_{S,1}|\Phi_{-1})\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([a,b] \times \mathbb{R}^3)} \end{aligned} \quad (94)$$

Note that we now replace the \lesssim symbol with some explicit constant \tilde{C} . From here, since \square^{-1} is the solution operator to the inhomogeneous wave equation on the interval $[0, T]$ with zero initial data as expressed in (15), we know from Theorem 18 that

$$\|\square^{-1}(|K_T|\Phi_{-1})\|_{L_t^\gamma L_x^{\frac{2}{1-\gamma}}([0,T] \times \mathbb{R}^3)} \leq C_\gamma \| |K_T|\Phi_{-1} \|_{L_t^{\frac{2}{1-\gamma}} L_x^{\frac{2}{2-\gamma}}([0,T] \times \mathbb{R}^3)} \quad (95)$$

and similarly for $|K_0|$ and $|K_{S,2}|$. Next, since $\|\Phi_{-1}\|_{L_t^2 L_x^2([0,T)\times\mathbb{R}^3)} \lesssim 1$ by (91) and $\frac{2}{1-\gamma} = 4+$ by the assumption that $\gamma = \frac{1}{2}+$, we can apply Hölder's inequality and (90) to (95) to get:

$$\begin{aligned} \| |K_T| \Phi_{-1} \|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}([0,T)\times\mathbb{R}^3)} &\leq \| |K_T| \|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \|\Phi_{-1}\|_{L_t^2 L_x^2([0,T)\times\mathbb{R}^3)} \\ &\lesssim \| |K_T| \|_{L_t^{\frac{2}{\gamma}} L_x^{4+}([0,T)\times\mathbb{R}^3)} \\ &\lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \end{aligned}$$

We obtain the same bound for the $|K_{S,2}|$ term. The $|K_0|$ term can be bounded by a constant since $|K_0|$ depends only on the initial data of the system. Summarizing, there exists a constant C :

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \leq CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}) + C\|\square^{-1}(|K_{S,1}|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \quad (96)$$

Now, let us set $u = \square^{-1}(|K_{S,1}|\Phi_{-1})$ for convenience of notation. Then, by Strichartz estimates and Hölder's inequality, we have the following fact:

$$\begin{aligned} \|u\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} &\leq C_\gamma \left(\|u(a)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(a)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \right. \\ &\quad \left. + \| |K_{S,1}| \|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \|\Phi_{-1}\|_{L_t^2 L_x^2([0,T)\times\mathbb{R}^3)} \right). \quad (97) \end{aligned}$$

Next, due to (91), we can choose a partition $0 = T_0 < T_1 < T_2 < \dots < T_N = T$ of the interval $[0, T]$ such that:

$$\|\Phi_{-1}\|_{L_t^2 L_x^2([T_j, T_{j+1}]\times\mathbb{R}^3)} \leq \frac{1}{2CC_\gamma} \quad (98)$$

for $j = 0, 1, \dots, N-1$.

Hence, by (97) and (96), we obtain:

$$\begin{aligned} \|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_j, T_{j+1}]\times\mathbb{R}^3)} &\leq CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} + \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)}) \\ &\quad + \frac{1}{2} \| |K_{S,1}| \|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_j, T_{j+1}]\times\mathbb{R}^3)}. \quad (99) \end{aligned}$$

This implies that for any $j = 0, 1, \dots, N-1$, we have the inequality:

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_j, T_{j+1}]\times\mathbb{R}^3)} \leq 2CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} + \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)}). \quad (100)$$

Using Strichartz estimates again, we obtain the bound:

$$\begin{aligned} \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} &\leq C_\gamma \left(\|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \right. \\ &\quad \left. + \| |K_{S,1}| \|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_{j-1}, T_j]\times\mathbb{R}^3)} \|\Phi_{-1}\|_{L_t^2 L_x^2([T_{j-1}, T_j]\times\mathbb{R}^3)} \right). \quad (101) \end{aligned}$$

We now apply Hölder's inequality and the bound (98) to (101) to get that:

$$\begin{aligned} \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} &\leq C_\gamma \left(\|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} + \frac{1}{2CC_\gamma} \| |K_{S,1}| \|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_{j-1}, T_j]\times\mathbb{R}^3)} \right). \quad (102) \end{aligned}$$

Next, by the estimate (100), we obtain that:

$$\begin{aligned} \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} &\leq C_\gamma \left(\|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \right) + \frac{1}{2C} \left(2CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \right. \\ &\quad \left. + \|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)}) \right). \quad (103) \end{aligned}$$

Finally, it follows that:

$$\begin{aligned} \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \\ \leq 2C_\gamma \left(\|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \right) + C_\gamma (1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}). \end{aligned} \quad (104)$$

Notice that $u(0) = \partial_t u(0) = 0$. Thus, performing an iteration of the above estimate (104), we obtain for any $k \in \{0, 1, \dots, N-1\}$:

$$\|u(T_k)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_k)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \leq \sum_{j=0}^{k-1} (2C_\gamma)^{k-1-j} (1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}).$$

Hence by (100), it follows that:

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_k, T_{k+1}] \times \mathbb{R}^3)} \leq 2CC_\gamma \left(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} + \sum_{j=0}^{k-1} (2C_\gamma)^{k-1-j} (1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}) \right). \quad (105)$$

Using the triangle inequality and summing (105) over $k = 0, 1, \dots, N-1$ and noting that N is some finite positive integer depending on $\|\Phi_{-1}\|_{L_t^\infty L_x^2}$, we get

$$\begin{aligned} \|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([0,T] \times \mathbb{R}^3)} &\leq \sum_{k=0}^{N-1} 2CC_\gamma \left(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} + \sum_{j=0}^{k-1} (2C_\gamma)^{k-1-j} (1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}) \right) \\ &\lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}. \end{aligned}$$

Since $\frac{2}{1-\gamma} = 4+$, we obtain the desired estimate (92). \square

8 Pallard's Decomposition and Bounding P(T)

In this section, we first recall the decomposition method in [11] and then apply the above estimates to gain a bound on the size of the momentum support of f , which we will denote by:

$$P(T) \stackrel{\text{def}}{=} 1 + \sup\{p \in \mathbb{R}^3 | \exists (t, x) \in [0, T] \times \mathbb{R}^3 \text{ such that } f(t, x, p) \neq 0\} \quad (106)$$

By the method of characteristics:

$$\frac{dV}{ds}(s; t, x, p') = E(s, X(s; t, x, p')) + \hat{V}(s; t, x, p') \times B(s, X(s; t, x, p')) \quad (107)$$

Taking the Euclidean inner product with $\hat{V}(s; t, x, p')$ on both sides and then integrating in time, we obtain:

$$\sqrt{1 + |V(s; t, x, p')|^2} = \sqrt{1 + |V(0; t, x, p')|^2} + \int_0^T E(s, X(s; t, x, p')) \cdot \hat{V}(s; t, x, p') ds \quad (108)$$

First, for $i = 1, 2, 3$ and $K_j = E_j + (\hat{p} \times B)_j$, we can decompose the electric field:

$$\begin{aligned} E_i(t, x) &= E_i^{(0)}(x) + \int_{\mathbb{R}^3} \left(\frac{(1 - |\hat{p}|^2)(x_i - t\hat{p})}{(t - \hat{p} \cdot x)^2} \right) Y \star_{t,x} (f \chi_{t \geq 0}) dp \\ &\quad - \sum_{j=1}^3 \int_{\mathbb{R}^3} \left(\frac{[t(t - \hat{p} \cdot x)(\hat{p}_i \hat{p}_j - e_i) + (x_i - t\hat{p}_i)(x_j - (\hat{p} \cdot x)\hat{p}_j)]}{p_0(t - \hat{p} \cdot x)^2} \right) \star_{t,x} (K_j f \chi_{t \geq 0}) dp \\ &\stackrel{\text{def}}{=} E_i^{(0)}(x) + F_i(t, x) + G_i(t, x) \end{aligned} \quad (109)$$

where e_i is the unit vector with all entries equal to 0 except for the i^{th} entry which is equal to 1. Also, the double convolution $\star_{t,x}$ is a binary operation defined by:

$$f_1 \star_{t,x} f_2 = \int_{\mathbb{R} \times \mathbb{R}^3} f_1(t - s, x - y) f_2(s, y) ds dy \quad (110)$$

and

$$Y \stackrel{\text{def}}{=} (4\pi t)^{-1} \delta_{|x|=t} \quad (111)$$

Following the scheme of [11], we can decompose the characteristic integral of the electric field into:

$$\int_0^T E(s, X(s; t, x, p')) \cdot \hat{V}(s; t, x, p') ds = I_0 + I_F + I_G \quad (112)$$

where I_0 depends only on the initial data term $E^{(0)}$ and

$$I_F \stackrel{\text{def}}{=} \int_0^T F(s, X(s; t, x, p')) \cdot \hat{V}(s; t, x, p') ds \quad (113)$$

and

$$I_G \stackrel{\text{def}}{=} \int_0^T G(s, X(s; t, x, p')) \cdot \hat{V}(s; t, x, p') ds \quad (114)$$

Pallard then bounds I_G by:

$$|I_G| \lesssim \int_0^T \int_s^T \int_{|y|=t-s} \int_{\mathbb{R}_p^3} \frac{(f|K|)(s, X(t) - y, p)}{p_0(1 - \hat{p} \cdot \omega)} \left(\sqrt{1 - \hat{V}(t) \cdot \omega} \right) dp \frac{d\sigma(y) dt}{4\pi|t-s|} ds \quad (115)$$

From here, Pallard [11] bounds the integral

$$\int_{\mathbb{R}_p^3} \frac{(f|K|)(s, X(t) - y, p)}{p_0(1 - \hat{p} \cdot \omega)} dp$$

using the term $m(t, x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} p_0 f(t, x, p) dp$. Instead, we preserve the singularity in the denominator:

$$|I_G| \lesssim \int_0^T \int_s^T \int_{|y|=t-s} (\sigma_{-1}|K|)(s, X(t) - y) \left(\sqrt{1 - \hat{V}(t) \cdot \omega} \right) \frac{d\sigma(y) dt}{4\pi|t-s|} ds \quad (116)$$

Split the integral into $I_G \lesssim I'_G + I''_G$ as follows:

$$\begin{aligned} |I_G| \lesssim & \int_0^T \int_s^{s+\epsilon(s)} \int_{|y|=t-s} (\sigma_{-1}|K|)(s, X(t) - y) \left(\sqrt{1 - \hat{V}(t) \cdot \omega} \right) \frac{d\sigma(y) dt}{4\pi|t-s|} ds \\ & + \int_0^T \int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)(s, X(t) - y) \left(\sqrt{1 - \hat{V}(t) \cdot \omega} \right) \frac{d\sigma(y) dt}{4\pi|t-s|} ds \end{aligned} \quad (117)$$

for

$$\epsilon(s) = \frac{T-s}{1+P(s)^8} \quad (118)$$

Note that the power of $P(s)$ in (118) is useful for bounding I'_G as in [11]. First, let us bound I''_G . By computing using Hölder's inequality as in [11]:

$$\begin{aligned} |I''_G| \lesssim & \int_0^T \left| \int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t) - y) (1 - \hat{V}(t) \cdot \omega) d\sigma(y) dt \right|^{\frac{2}{3}} \\ & \times \left(\int_{s+\epsilon(s)}^T \int_{|y|=t-s} ((1 - \hat{V}(t) \cdot \omega)^{-\frac{1}{6}})^3 d\sigma dt \right)^{\frac{1}{3}} ds \\ \lesssim & \int_0^T \left| \int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t) - y) (1 - \hat{V}(t) \cdot \omega) d\sigma(y) dt \right|^{\frac{2}{3}} \ln^{\frac{1}{3}} \left(\frac{T-s}{\epsilon(s)} \right) ds \end{aligned} \quad (119)$$

Setting $\omega = \omega(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$:

$$\begin{aligned} & \int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t) - y) (1 - \hat{V}(t) \cdot \omega) d\sigma(y) dt \\ & = \int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t) - (t-s)\omega(\theta, \phi)) (1 - \hat{V}(t) \cdot \omega(\theta, \phi)) (t-s)^2 \sin \theta d\theta d\phi dt \end{aligned} \quad (120)$$

Consider the change of variables $\Psi : (s_1, s_2) \times (0, \pi) \times (0, 2\pi) \rightarrow \Psi((s_1, s_2) \times (0, \pi) \times (0, 2\pi))$ mapping

$$(t, \theta, \phi) \mapsto X(t) - (t - s)\omega(\theta, \phi) \stackrel{\text{def}}{=} z$$

The Jacobian of this map is $J = (\hat{V}(t) \cdot \omega - 1)(t - s)^2 \sin \theta$. Applying this change of variables to (120) and inserting our choice of $\epsilon(s)$, we obtain:

$$|I_G''| \lesssim \int_0^T \left| \int_{\Psi((s_1, s_2) \times (0, \pi) \times (0, 2\pi))} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, z) dz dt \right| \ln^{\frac{1}{3}} \left(1 + P(s) \right) ds \quad (121)$$

Following [11] precisely, we also know that $I_G' \lesssim 1$. (This is done through first applying Hölder's inequality to isolate the first term and then using conservation law $\|K\|_{L_t^\infty L_x^2} \lesssim 1$. Finally, by the definition of $\epsilon(s)$, the leftover integral is bounded.) Thus, we arrive at the estimate:

$$|I_G| \lesssim 1 + \|\sigma_{-1}|K| \ln^{\frac{1}{3}}(1 + P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0, T] \times \mathbb{R}^3)} \quad (122)$$

Next, we recognize that F is equivalent to our E_T term as expressed in (11). Thus, using the proof of Proposition 21:

$$|I_F| \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} \quad (123)$$

In conclusion:

Proposition 28. *By (122) and (123), we have the following bound for $P(T)$:*

$$|P(T)| \lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} + \|\sigma_{-1}|K| \ln^{\frac{1}{3}}(1 + P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0, T] \times \mathbb{R}^3)} \quad (124)$$

9 Moment Bounds

We conclude by applying the estimates given by (89), (90) and (92) on $|K|$ under the assumption that $\|\Phi_{-1}\|_{L_t^\infty L_x^2} \lesssim 1$:

$$\begin{aligned} \|\sigma_{-1}|K| \ln^{\frac{1}{3}}(1 + P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0, T] \times \mathbb{R}^3)} &\leq \ln^{\frac{1}{3}}(1 + P(T)) \| |K| \|_{L_t^1 L_x^{4+}} \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \\ &\lesssim \ln^{\frac{1}{3}}(1 + P(T)) (1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}) \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \end{aligned}$$

Notice that our choice of Hölder exponents used in the first line above allow for the Lebesgue norm exponents on both terms involving σ_{-1} to be approximately equivalent to $\frac{12}{5}$. This choice of Hölder exponents simplifies our computation. Other choices yield similar results. We can now use Lemma 15 to bound each term in (124) for some $\beta > 0$ arbitrarily small:

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \lesssim \|p_0^{\frac{24}{5r}-1} f\|_{L_t^\infty L_x^r L_p^1} \quad (125)$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \lesssim \|p_0^{\frac{24}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \quad (126)$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} \lesssim \|p_0^{\frac{6}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \quad (127)$$

We can extract $\frac{12}{10r} - \delta$ power of p_0 for some $\delta > 0$ arbitrarily small from each of (125) and (126) and $\frac{3}{r} - \delta$ power of p_0 from (127). Thus:

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \lesssim \|p_0^{\frac{18}{5r}-1} f\|_{L_t^\infty L_x^r L_p^1} P(T)^{\frac{1}{2}-} \quad (128)$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \lesssim \|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} P(T)^{\frac{1}{2}-} \quad (129)$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} \lesssim \|p_0^{\frac{3}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} P(T)^{1-} \quad (130)$$

where $P(T)^{1-}$ indicates a power of $P(T)$ smaller than 1 by an arbitrarily small amount. Assume that $\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$. (Hence $\|p_0^{\frac{3}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim \|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$.)

Plugging these into (124), we obtain the bound:

$$P(T) \lesssim 1 + \ln^{\frac{1}{3}}(1 + P(T))P(T)^{1-} \quad (131)$$

which implies that $P(T) \lesssim 1$ since $P(T) > 1$. Finally the last term we need to take care of is the assumption that $\|\Phi_{-1}\|_{L_t^\infty L_x^2} \lesssim 1$. By employing similar proof to Lemma 15, we see that:

Proposition 29. *Given $r \in [1, 2]$, we have the estimate:*

$$\|\Phi_{-1}\|_{L_x^2} \lesssim \|p_0^\alpha f\|_{L_x^r L_p^1}^{\frac{r}{2}} \quad (132)$$

where $\alpha > \frac{2}{r} - 1$.

Proof. Fix some $\omega \in \mathbb{S}^2$ and let $r = \frac{2}{q}$. Then $\frac{q'}{2} \geq 1$ (since $\frac{1}{q} + \frac{1}{q'} = 1$ and $q \geq 2$) and $\frac{1}{1+\hat{p} \cdot \omega} \lesssim p_0^2$ implies:

$$\int_{\mathbb{R}^3} \frac{f(t, x, p)}{p_0(1 + \hat{p} \cdot \omega)^{\frac{1}{2}}} dp \lesssim \left(\int_{\mathbb{R}^3} \frac{1}{p_0^{(\beta+1)q'}(1 + \hat{p} \cdot \omega)^{\frac{q'}{2}}} dp \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^3} p_0^{\beta q} f(t, x, p) dp \right)^{\frac{1}{q}} \|f\|_{L_{t,x,p}^\infty}^{\frac{q-1}{q}} \quad (133)$$

$$\lesssim \left(\int_{\mathbb{R}^3} \frac{1}{p_0^{\beta q' + 2}(1 + \hat{p} \cdot \omega)} dp \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^3} p_0^{\beta q} f(t, x, p) dp \right)^{\frac{1}{q}} \quad (134)$$

By (42) in the proof of Lemma 15, we know that the first integral on the right hand side is bounded by a constant when $\beta q' > 1$, i.e. $\beta q > q - 1$. Taking the L^2 norm of this inequality:

$$\left\| \int_{\mathbb{R}^3} \frac{f(t, x, p)}{p_0(1 + \hat{p} \cdot \omega)^{\frac{1}{2}}} dp \right\|_{L_x^2} \lesssim \left\| \left(\int_{\mathbb{R}^3} p_0^{\beta q} f(t, x, p) dp \right)^{\frac{1}{q}} \right\|_{L_x^2} = \left\| \left(\int_{\mathbb{R}^3} p_0^{\beta q} f(t, x, p) dp \right) \right\|_{L_x^{\frac{2}{q}} L_p^1}^{\frac{1}{q}} \quad (135)$$

Finally, setting $\alpha = \beta q$, we obtain that

$$\alpha > q - 1 = \frac{2}{r} - 1.$$

Taking the maximum over all $\omega \in \mathbb{S}^2$ retains the same upper bound. Hence, this completes the proof. \square

In particular, notice that for $1 \leq r \leq 2$:

$$\|\Phi_{-1}\|_{L_x^2} \lesssim \|p_0^{\frac{2}{r}-1+\beta} f\|_{L_x^r L_p^1} \lesssim \|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1 \quad (136)$$

Thus, if $\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$, then $\|\Phi_{-1}\|_{L_x^2} \lesssim 1$. Hence, all of the terms in (124), which implicitly included the assumption $\|\Phi_{-1}\|_{L_x^2} \lesssim 1$, are bounded. Thus, indeed we do know that $P(T) \lesssim 1$. Thus, we can extend our local solution on the time interval $[0, T)$ to a larger time interval $[0, T + \epsilon]$. This concludes the proof of

$$\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1 \quad (137)$$

as a continuation criteria for $1 \leq r \leq 2$.

10 Proof of Theorem 10

In this section, we prove our final result, Theorem 10. First, we state the following bounds analogous to [8]. The inequality (140) is proven analogously to Proposition 4.3 in [8], where we replace the fixed unit vector e_3 with the time-dependent unit vector $n_3(t)$. This change does not affect the proof because our inequality is pointwise in time. Before stating our main propositions, we define the following notation for vectors $v, w \in \mathbb{R}^3$:

$$\angle(v, \pm w) \stackrel{\text{def}}{=} \min\{\angle(v, w), \angle(v, -w)\},$$

which will be used throughout this section.

Proposition 30. *For any $p \in \mathbb{R}^3$ and $\omega \in \mathbb{S}^2$:*

$$(1 + \hat{p} \cdot \omega)^{-1} \lesssim \min\{p_0^2, (\angle(\frac{p}{|p|}, -\omega))^{-2}\} \quad (138)$$

Further, if $\gamma = \tan^{-1}(\frac{p \cdot n_2(t)}{p \cdot n_1(t)})$ and $p \in \text{supp}\{f\}$, then

$$|p| \lesssim \frac{\kappa(t, \gamma(p))}{\angle(\frac{p}{|p|}, \pm n_3(t))} \quad (139)$$

Combining (138) and (139), we obtain the following estimate for $p \in \text{supp}\{f\}$:

$$(1 + \hat{p} \cdot \omega)^{-1} \lesssim \min\left\{\left(\frac{\kappa(t, \gamma(p))}{\angle(\frac{p}{|p|}, \pm n_3(t))}\right)^2, (\angle(\frac{p}{|p|}, -\omega))^{-2}\right\} \quad (140)$$

Define $\omega^{(i)} = (\sin(\theta^{(i)}) \cos(\phi^{(i)}), \sin(\theta^{(i)}) \sin(\phi^{(i)}), \cos(\theta^{(i)}))$ where $\omega^{(i)}$ is the transformation of ω under a rotation matrix that takes e_i to $n_i(T_i)$. Thus, $\theta^{(i)} = \angle(n_3(T_i), \omega^{(i)})$. By similar arguments to Proposition 4.4 in [8], we obtain:

Proposition 31. *We have the uniform estimate*

$$\int_{\mathbb{R}^3} \frac{f(s, x + r\omega^{(i)}, p)}{p_0(1 + \hat{p} \cdot \omega^{(i)})} dp \lesssim \min\{P(s)^2 \log(P(s)), \frac{A(s)^4 \log(P(s))}{(\angle(n_3(s), \pm \omega^{(i)}))^2}\} \quad (141)$$

for $s \in [T_i, T_{i+1})$.

Proof. We follow the proof of Proposition 4.4 in [8], emphasizing the steps in which we deviate from their proof. First, pick spherical coordinates $\theta_{(i)}, \phi_{(i)}$ such that $-\omega^{(i)}$ lies on the half-axis $\theta_{(i)} = 0$. Then, by (138), we obtain the estimate

$$(1 + p \cdot \omega^{(i)})^{-1} \lesssim \min\{p_0^2, (\theta_{(i)})^{-2}\}. \quad (142)$$

By the definition of $P(s)$, the particle density $f(s, x + r\omega^{(i)}, p) = 0$ for $|p| > P(s)$. Thus, the conservation law $\|f\|_{L_{x,p}^\infty} \lesssim 1$ and the inequality (142) imply that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{f(s, x + r\omega^{(i)}, p)}{p_0(1 + \hat{p} \cdot \omega^{(i)})} dp &\lesssim \int_{|p| \leq P(s)} \frac{1}{p_0(1 + \hat{p} \cdot \omega^{(i)})} dp \\ &\lesssim \int_0^{P(s)} \int_0^\pi \int_0^{2\pi} \frac{1}{p_0(1 + \hat{p} \cdot \omega^{(i)})} d|p| d\theta_{(i)} d\phi_{(i)} \\ &\lesssim \int_0^{P(s)} \int_0^{P(s)^{-1}} p_0^2 d|p| d\theta_{(i)} + \int_0^{P(s)} \int_{P(s)^{-1}}^\pi (\theta_{(i)})^{-2} d|p| d\theta_{(i)} \\ &\lesssim P(s)^2 \log(P(s)), \end{aligned}$$

which proves the first part of our proposition. We now move on to prove the second bound we need. Let $\beta_i = \angle(n_3(s), \pm \omega^{(i)})$. We partition the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of β_i as in [8]:

$$I_i = \{\angle(n_3(s), \omega^{(i)}) \leq \frac{\beta_i}{2}\} \cup I_i = \{\angle(n_3(s), -\omega^{(i)}) \leq \frac{\beta_i}{2}\}$$

$$II_i = \{\angle(n_3(s), \pm\omega^{(i)}) \geq \frac{\beta_i}{2}\} \cap I_i = \{\angle(n_3(s), \pm\omega^{(i)}) \geq \frac{\beta_i}{2}\}.$$

By the definition of β_i and the triangle inequality:

$$\angle\left(\frac{p}{|p|}, \omega^{(i)}\right) \geq |\angle(n_3(s), \omega^{(i)}) - \angle\left(\frac{p}{|p|}, n_3(s)\right)|$$

if $\angle(n_3(s), \omega^{(i)}) \leq \frac{\beta}{2}$. Similarly, if $\angle(n_3(s), -\omega^{(i)}) \leq \frac{\beta}{2}$, then

$$\angle\left(\frac{p}{|p|}, \omega^{(i)}\right) \geq |\angle(n_3(s), \omega^{(i)}) - \angle\left(\frac{p}{|p|}, n_3(s)\right)|.$$

We can do the same estimate for $\angle(\frac{p}{|p|}, -\omega^{(i)})$, and hence,

$$\angle(n_3(s), \pm\omega^{(i)}) \leq \frac{\beta}{2}.$$

By (140), we now know that

$$(1 + p \cdot \omega^{(i)})^{-1} \lesssim \beta^{-2}.$$

Using this estimate for region I_i and defining the domains D_i and \tilde{D}_i as

$$D_i = \{p \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3 \text{ such that } f(s, x, p) \neq 0\}$$

and

$$D_i = \{(p_1, p_2) \in \mathbb{R}^2 \mid \exists x \in \mathbb{R}^3, p_3 \in \mathbb{R} \text{ such that } f(s, x, p_1, p_2, p_3) \neq 0\},$$

we obtain the following estimate on region I_i :

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{f(s, x + r\omega^{(i)}, p)}{(1 + p \cdot \omega^{(i)})^{-1}} dp &\lesssim \beta^{-2} \int_{D_i} \frac{1}{p_0} dp \\ &\lesssim \beta^{-2} \int_{\tilde{D}_i} \int_{-P(s)}^{P(s)} \frac{1}{\sqrt{1 + p_3}} dp_3 dp_1 dp_2 \\ &\lesssim \beta^{-2} \log(P(s)) \int_{\tilde{D}_i} dp_1 dp_2 \\ &\lesssim \beta^{-2} \log(P(s)) \int_0^{2\pi} \int_0^{\kappa(s, \gamma)} u du d\gamma \\ &\lesssim \beta^{-2} \log(P(s)) \|\kappa(s, \gamma)\|_{L^4_\gamma}^2 \end{aligned}$$

(In the above, we used polar coordinate to compute the integral over \tilde{D} and Hölder's inequality in γ in the last step.) Thus, we have obtained the bound in region I_i :

$$\int_{I_i} \frac{f(s, x + r\omega^{(i)}, p)}{p_0(1 + p \cdot \omega^{(i)})} dp \lesssim \beta^{-2} \log(P(s)) A(s)^2 \lesssim \frac{\log(P(s)) A(s)^4}{(\angle(n_3(s), \pm\omega^{(i)}))^2}.$$

For region II_i , pick a system of polar coordinates (θ_s, ϕ_s) such that $p \cdot n_3(s) = |p| \cos(\theta_s)$, i.e. $\theta_s = \angle(p, n_3(s))$. Hence, by definition of β , we have that $\frac{\beta}{2} \leq \theta_s \leq \frac{\pi}{2} - \frac{\beta}{2}$ and by definition of $\gamma = \gamma(p)$, we also have that $\phi_s = \gamma(p)$. By (139), we have that

$$|p| \lesssim \kappa(t, \phi_s)(\theta_s^{-1} + (\pi - \theta_s)^{-1}).$$

Using (140), we obtain

$$\begin{aligned} \int_{II_i} \frac{f(s, x + r\omega^{(i)}, p)}{p_0(1 + p \cdot \omega^{(i)})} dp &\lesssim \int_0^{2\pi} d\phi_s \int_{\frac{\beta}{2}}^{\frac{\pi}{2} - \frac{\beta}{2}} \sin(\theta_s) d\theta_s \int_0^{C\kappa(t, \phi_s)(\theta_s^{-1} + (\pi - \theta_s)^{-1})} |p| \kappa(t, \phi_s)^2 (\theta_s^{-2} + (\pi - \theta_s)^{-2}) d|p| \\ &\lesssim \beta^{-2} A(t)^4 \lesssim \frac{\log(P(s)) A(s)^4}{(\angle(n_3(s), \pm\omega^{(i)}))^2} \end{aligned}$$

Summing the integrals over the domains I_i and II_i , we obtain the second bound we wanted. This completes our proof. \square

In the above, $\angle(n_3(s), \pm\omega^{(i)}) \stackrel{\text{def}}{=} \min\{\angle(n_3(s), \omega^{(i)}), \angle(n_3(s), -\omega^{(i)})\}$. Notice that the above inequality is pointwise in time. Thus, the proof Proposition 31 differs from the proof of Proposition 4.4 in [8] only in that we replace the unit vector $e_3 = (0, 0, 1)$ with $n_3(s)$ and ω with $\omega^{(i)}$. We now give modified arguments for momentum support on planes changing uniformly continuously in time.

Proposition 32. *For $t \in [0, T]$:*

$$|E_T(t, x)| + |B_T(t, x)| \lesssim \log(P(t)) + (\log(P(t)))^2 \int_0^t A(s)^4 ds \quad (143)$$

Proof. Using the bound (11) and partitioning the time interval

$$[0, t] = \cup_{0 \leq n_t} ([T_i, T_{i+1}] \cap [0, t])$$

as given by the conditions (28) and (29):

$$\begin{aligned} |E_T(t, x)| + |B_T(t, x)| &\lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{(t-s)^2 p_0^2 (1 + \hat{p} \cdot \omega)^{\frac{3}{2}}} dp d\omega \\ &= \int_0^t \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{p_0^2 (1 + \hat{p} \cdot \omega)^{\frac{3}{2}}} dp \sin(\theta) d\theta d\phi ds \\ &= \sum_0^{n_t} \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_0^\pi \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega^{(i)}, p)}{p_0^2 (1 + \hat{p} \cdot \omega^{(i)})^{\frac{3}{2}}} dp \sin(\theta^{(i)}) d\theta^{(i)} d\phi^{(i)} ds \end{aligned}$$

We can divide the integral over $d\theta^{(i)}$ into three regions:

$$\int_0^\pi \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega^{(i)}, p)}{p_0^2 (1 + \hat{p} \cdot \omega^{(i)})^{\frac{3}{2}}} dp d\theta^{(i)} = A_i + B_i + C_i$$

where A_i is the integral over $[0, P(t)^{-1}]$, B_i is the integral over $[P(t)^{-1}, \pi - P(t)^{-1}]$, and C_i is the integral over $[\pi - P(t)^{-1}, \pi]$. We estimate each of these integrals using Proposition 31.

$$\begin{aligned} B_i &= \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega^{(i)}, p)}{p_0^2 (1 + \hat{p} \cdot \omega^{(i)})^{\frac{3}{2}}} dp \sin(\theta^{(i)}) d\theta^{(i)} \\ &\lesssim \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^4 \log(P(s))}{(\angle(n_3(s), \pm\omega^{(i)}))^2} \sin(\theta^{(i)}) d\theta^{(i)} \\ &\lesssim \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^4 \log(P(s))}{(\theta^{(i)})^2} \sin(\theta^{(i)}) d\theta^{(i)} + \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^4 \log(P(s))}{(\pi - \theta^{(i)})^2} \sin(\pi - \theta^{(i)}) d\theta^{(i)} \end{aligned}$$

where in the third line we used the fact that $\sin(\theta^{(i)}) = \sin(\pi - \theta^{(i)})$ and we also used the following triangle inequality argument for angles:

$$\angle(n_3(s), \pm\omega^{(i)}) \geq |\angle(n_3(T_i), \pm\omega^{(i)}) - \angle(n_3(s), n_3(T_i))|$$

In the time interval $[T_i, T_{i+1}]$, we have that $\angle(n_3(s), n_3(T_i)) < \frac{P(t)^{-1}}{4}$. Further, we are integrating over the interval $\theta^{(i)} = \angle(n_3(T_i), \omega^{(i)}) \in [P(t)^{-1}, \pi - P(t)^{-1}]$ and $\pi - \theta^{(i)} = \angle(n_3(T_i), -\omega^{(i)}) \in [P(t)^{-1}, \pi - P(t)^{-1}]$. Thus,

$$|\angle(n_3(T_i), \omega^{(i)}) - \angle(n_3(s), n_3(T_i))| \approx \theta^{(i)}$$

and

$$|\angle(n_3(T_i), -\omega^{(i)}) - \angle(n_3(s), n_3(T_i))| \approx \pi - \theta^{(i)}$$

Evaluating the integral, we obtain:

$$B_i \lesssim A(s)^4 \log(P(t))^2$$

and

$$\sum_0^{n_t} \int_{T_i}^{T_{i+1}} B_i ds \lesssim \log(P(t))^2 \int_0^t A(s)^4 ds$$

Next, evaluating A_i and C_i using the estimate

$$\int \frac{f(s, x + r\omega, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \lesssim P(s)^2 \log(P(s))$$

we obtain that

$$\begin{aligned} A_i &\lesssim \int_0^{P(t)^{-1}} P(s)^2 \log(P(s)) \sin(\theta^{(i)}) d\theta^{(i)} \\ &\lesssim \log(P(t)) \end{aligned}$$

and similarly for $C_i \lesssim \log(P(t))$. Summing over $i = 1, \dots, n_t$, we obtain our result. \square

Next, we bound the $E_{S,1} + B_{S,1}$ term. To do so, we apply the argument directly from [8]:

Proposition 33. For $s \in [0, T)$:

$$\| \int_{\mathbb{R}^3} f(s, x, p) dp \|_{L_x^\infty} \lesssim A(s)^2 P(s) \quad (144)$$

Proof. Consider coordinates on \mathbb{R}^3 such that $Q(s)$ lies in the $(p_1, p_2, 0)$ plane. By the support of f and since f is a bounded function,

$$\| \int_{\mathbb{R}^3} f dp \|_{L^\infty} \lesssim \int_{-P(s)}^{P(s)} dp_3 \int_0^{2\pi} d\gamma \int_0^{\kappa(s, \gamma)} r dr \lesssim A(s)^2 P(s)$$

\square

Since we still have the same bound (144) as in [8], the proof of Proposition 5.3 in [8] follows exactly:

Proposition 34. For $t \in [0, T)$:

$$\int_0^t |E_{S,1}| + |B_{S,1}| ds \lesssim \sqrt{\log P(t)} \int_0^t A(s)^2 P(s) ds \quad (145)$$

Finally, we have:

Proposition 35. For $t \in [0, T)$:

$$|E_{S,2}| + |B_{S,2}| \lesssim P(t) \log P(t) + P(t) \log P(t) \left(\int_0^t A(s)^8 ds \right)^{\frac{1}{2}} \quad (146)$$

Proof. Applying Hölder's inequality to (31):

$$|E_{S,2}| + |B_{S,2}| \lesssim \|K_g\|_{L^2(C_{t,x})} \left(\int_0^t \int_0^{2\pi} \int_0^\pi \left(\int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \right)^2 \sin \theta d\theta d\phi ds \right)^{\frac{1}{2}} \quad (147)$$

The $\|K_g\|_{L^2(C_{t,x})}$ term is uniformly bounded so we just have to get an estimate on the second term on the right. We apply the same decomposition as in the proof of Proposition 32. First, we split the integral over θ into three intervals and apply (31) to the momentum integral to obtain the inequality:

$$\int_0^t \int_0^{2\pi} \int_0^\pi \left(\int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \right)^2 \sin \theta d\theta d\phi ds \lesssim \sum_{i=0}^{n_t} A_i + B_i + C_i \quad (148)$$

where

$$\begin{aligned} A_i &= \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_0^{P(t)^{-1}} P(s)^4 \log(P(s))^2 \sin \theta d\theta d\phi ds \\ B_i &= \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^8 \log(P(s))^2}{(\mathcal{L}(n_3(s), \pm \omega^{(i)})^4)} \sin \theta d\theta d\phi ds \end{aligned}$$

$$C_i = \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_{\pi-P(t)^{-1}}^{\pi} P(s)^4 \log(P(s))^2 \sin(\pi - \theta) d\theta d\phi ds$$

Now, we apply the same methods to bound A_i , B_i and C_i as in Proposition 32 to obtain that:

$$\sum_{i=0}^{n_t} A_i + B_i + C_i \lesssim P(t)^2 \log(P(t))^2 + P(t)^2 \log(P(t))^2 \int_0^t A(s)^8 ds \quad (149)$$

Plugging (149) into (147), we obtain our result. \square

Notice that we have proven the same bounds on the fields E and B as found in [8]. Thus, we can borrow the same proof from Proposition 6.1 in [8] to obtain that $P(T) \lesssim 1$. Hence, by Theorem 2, we can extend our solution (f, E, B) to a larger time interval $[0, T + \epsilon]$.

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